OSCILLATION OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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SARI

Tulisan ini membahas persamaan diferensial nonlinear orde dua yang berbentuk:

$$u'' + f(t, u) = 0, \quad t \geq 0$$

Kriteria oskilasi untuk persamaan di atas dikaji dengan memodifikasi metode yang pernah dilakukan untuk menentukan kriteria oskilasi untuk persamaan diferensial di atas. Hasil yang diperoleh memuat dan mengembangkan kriteria oskilasi sebelumnya. Beberapa asumsi dalam teorema lebih lemah dibandingkan dengan yang digunakan sebelumnya.

ABSTRACT

In this paper we consider the following second order nonlinear differential equations:

$$u'' + f(t, u) = 0, \quad t \geq 0$$

Oscillation criteria for the above equation will be established by modification of the method that has been used previously. The results obtained will contain and improve the previous results. Some conditions imposed in the theorem will be less weaker than used before.

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1. INTRODUCTION

We consider the second order nonlinear differential equation

$$u''+f(t,u)=0,\quad t\geq 0 \tag{1.1}$$

where the function f satisfies certain conditions to be given later.

Definition 1.1

(a) A solution $u(t)$ of *(1.1)* is called oscillatory in $[0,\infty)$ if for every $t_0\in[0,\infty)$ there exists a $t_1\geq t_0$ such that $u(t_1)=0$.

(b) A solution $u(t)$ of (1) is called non oscillatory in $[0,\infty)$ if there exists a $t_1\geq 0$ such that $u(t)\neq 0$ for $t\geq t_1$.

(c) The differential equation (1.1) is called oscillatory $[0,\infty)$ if every solution of (1.1) is oscillatory in $[0,\infty)$.

Oscillation criteria for the differential equation

$$u''+a(t)f(u)=0,\quad t\geq 0 \tag{1.2}$$

including the Emden-Fowler equation, where $f(u)=|u|^{\gamma}\,sgn\,u$, has been developed by many authors, in particular, the papers of [2,3,6,7,8,9]. Oscillation criteria for second order nonlinear inequalities which contain Eqn (1.1) has been developed in [4]. However, the conditions for function $f(t,u)$ in this paper are quite different to conditions in [4] and also the oscillation criteria. In [1,5] the oscillation criteria for Eqn (1.1) has been discussed. Some conditions in this paper are less weaker than those in [1] and our results are another type of oscillatory criteria for Eqn (1.1).

Our main results will contain and extend some previous results in [2,5].Oscillation criteria will be established by modifying the methods that have been used previously in [1] and [5].

2. OSCILLATION THEOREMS

The following assumptions on function f will be retained in the sequel.

Assumption A There exist a function $a\in C[0,\infty)$ and a function $\phi\in C^1(-\infty,\infty)$ such that

$$\frac{f(t,u)}{\phi(u)}\geq a(t),\ t\geq 0,$$

with $\phi'>0, u\phi(u)>0$ for $u\neq 0$ and

$$G(u)=\int_0^u \frac{d\tau}{\phi(\tau)}<\infty .$$

We now have the following theorems.

Theorem 2.1 Assume that Assumption A holds and there exists a function $\rho \in C^1[0,\infty)$ with $\rho > 0$ and $\rho' \le 0$ such that

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \rho^{\lambda}(t)\int_0^t a(\tau)d\tau dt = \infty \tag{2.1}$$

for some $\lambda \ge 0$. Then Eqn (1.1) is oscillatory in $[0,\infty)$.

Proof Suppose that Eqn (1.1) has a solution $u(t)$ which is non oscillatory in $[0,\infty)$. Assume that $u(t) > 0$ on $[t_0,\infty)$ for some $t_0 \ge 0$. Let

$$v(t)=\frac{\rho^{\lambda}(t)u'(t)}{\phi(u(t))},\ t\ge t_0$$

Then

$$v'=\frac{\rho^{\lambda}u''+\lambda\rho^{\lambda-1}\rho' u'}{\phi(u)}-\rho^{\lambda}\frac{u'^2\phi'(u)}{\phi^2(u)}$$

$$=-\rho^{\lambda}\frac{f(t,u)}{\phi(u)}+\frac{\lambda\rho' v}{\rho}-\frac{v^2}{\rho^{\lambda}}\phi'(u)$$

By Assumption A and $\rho > 0$, we obtain

$$v' \le -\rho^{\lambda}(t)a(t)+\lambda\frac{\rho'}{\rho}v,\ t\ge t_0$$

$$\frac{\rho v'-\lambda' v}{\rho}\le -\rho^{\lambda}(t)a(t)$$

$$\frac{\rho v'-\lambda' v}{\rho^{\lambda+1}}\le -a(t)$$

$$\frac{d}{dt}\left(\frac{v}{\rho^{\lambda}}\right)\le -a(t),\ t\ge t_0 \tag{2.2}$$

Integrating on $[t_0,t]$, we have

$$\frac{v(t)}{\rho^{\lambda}(t)} \le \frac{v(t_0)}{\rho^{\lambda}(t_0)} - \int_{t_o}^{t} a(\tau)d\tau$$

Let $v(t_0) > 0$, in case $v(t_0) \le 0$, we omit $\dfrac{v(t_0)}{\rho^{\lambda}(t_0)}$ from above equation.

Multiply by $\rho^{\lambda}(t) > 0$, we obtain

$$v(t) \le \frac{v(t_0)}{\rho^{\lambda}(t_0)}\rho^{\lambda}(t) - \rho^{\lambda}(t)\int_{t_0}^{t} a(\tau)d\tau$$

$$\frac{\rho^{\lambda}(t)u'(t)}{\phi(u(t))} \le \frac{v(t_0)}{\rho^{\lambda}(t_0)}\rho^{\lambda}(t) - \rho^{\lambda}(t)\int_{t_0}^{t} a(\tau)d\tau$$

Integrating on $[t_0,T]$, we obtain

$$\int_{t_0}^{T}\frac{\rho^{\lambda}(t)u'(t)}{\phi(u(t))}dt \le \frac{v(t_0)}{\rho^{\lambda}(t_0)}\int_{t_0}^{T}\rho^{\lambda}(t)dt - \int_{t_0}^{T}\rho^{\lambda}(t)\int_{t_0}^{t} a(\tau)d\tau dt.$$

Since $\rho(t)$ decreasing on $[0,\infty)$, $\rho > 0$ and $\lambda \ge 0$, we get

$$\int_{t_0}^{T}\rho^{\lambda}(t)dG(u(t)) \le \frac{v(t_0)}{\rho^{\lambda}(t_0)}\rho^{\lambda}(t_0)(T-t_0) - \int_{t_0}^{T}\rho^{\lambda}(t)\int_{t_0}^{t} a(\tau)d\tau dt$$

$$\rho^{\lambda}(t)G(u(t)) - \rho^{\lambda}(t_0)G(u(t_0)) - \lambda\int_{t_0}^{T} G(u(t))\rho^{\lambda-1}(t)\rho'(t)dt \le$$

$$\le v(t_0)(T-t_0) - \int_{t_0}^{T}\rho^{\lambda}(t)\int_{t_0}^{t} a(\tau)d\tau dt$$

Since $\rho^{\lambda}(T) > 0, G(u(T) \ge 0,$ and $\rho'(t) \le 0, \lambda \ge 0,$ we obtain

$$\int_{0}^{T}\rho^{\lambda}(t)\int_{0}^{t} a(\tau)d\tau dt \le v(t_0)(T-t_0) + C, \quad \text{where} \quad C = \rho^{\lambda}(t_0)G(u(t_0)).$$

Dividing by T, we have

$$\frac{1}{T}\int_{t_0}^{T}\rho^{\lambda}(t)\int_{t_0}^{t} a(\tau)d\tau dt \le v(t_0)\left(1-\frac{t_0}{T}\right) + \frac{C}{T}.$$

Condition (2.1) gives a contradiction. Similar condition will be obtained when $u(t) < 0$ on $[0,\infty)$.

Remark In case $\rho = 1$, or $\lambda = 0$, we obtain the oscillation criteria in [2]. When $\lambda = 1$, we obtain the oscillation criteria in [6].The condition for ρ is less weaker than in [1] and condition (2.1) is another type of oscillation criteria for Eqn (1.1) .

Theorem 2.2 Assume that Assumption A holds and there exists a function $\rho \in C^2[0,\infty)$ with $\rho>0$, $\rho' \leq 0$ and $\rho'' \geq 0$ such that

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T\int_0^t \rho^\lambda(\tau)a(\tau)d\tau dt = \infty, \tag{2.3}$$

for some $\lambda \geq 1$. Then Eqn (1.1) is oscillatory in $[0,\infty)$.

Proof Suppose that u (t) is a non oscillatory solution of Eqn (1.1) in $[0,\infty)$. Let u (t) > 0 on $[t_0,\infty)$ for some $t_0 \geq 0$. Let

$$v(t) = \frac{\rho^\lambda(t)u'(t)}{\phi(u(t))},\ t \geq t_0$$

Arguing as in the proof of theorem 2.1, we obtain the eqn (2.2), i.e

$$\frac{d}{dt}\left(\frac{v}{\rho^\lambda}\right) \leq -a(t)$$

$$\rho^\lambda \frac{d}{dt}\left(\frac{v}{\rho^\lambda}\right) \leq -\rho^\lambda(t)a(t) \tag{2.4}$$

Integrating (2.4) on $[t_0, t]$, we get

$$\int_{t_0}^t \rho^\lambda(\tau)d\left(\frac{v}{\rho^\lambda}\right) \leq -\int_{t_0}^t \rho^\lambda(\tau)a(\tau)d\tau$$

$$v(t)-v(t_0)-\lambda\int_{t_0}^t \frac{v(\tau)\rho'(\tau)}{\rho}d\tau \leq -\int_{t_0}^t \rho^\lambda(\tau)a(\tau)d\tau$$

$$v(t) \leq v(t_0)+\lambda\int_{t_0}^t \rho^{\lambda-1}(\tau)\rho'(\rho)dG(u(\tau)) - \int_{t_0}^t \rho^\lambda(\tau)a(\tau)d\tau$$

$$v(t) \leq v(t_0)+\lambda[\rho^{\lambda-1}(t)\rho'(t)G(u(t)) - \rho^{\lambda-1}(t_0)\rho'(t_0))G(u(t_0))] -$$
$$-\lambda\int_{t_0}^t G(u(\tau))[\rho^{\lambda-1}(\tau)\rho''(\tau)+(\lambda-1)\rho^{\lambda-2}(\tau)(\rho'(\tau))^2]d\tau -$$
$$\int_{t_0}^t \rho^\lambda(\tau)a(\tau)d\tau.$$

Since $\rho > 0$, $\rho' \leq 0$, $\rho'' \geq 0$, $G(u(t)) \geq 0$ and $\lambda \geq 1$, we have

$$v(t) \leq C_1 - \int_{t_o}^{t} \rho^{\lambda}(\tau) a(\tau) d\tau$$

where
$$C_1 = v(t_0) - \lambda \rho^{\lambda-1}(t_0)\rho'(t_0) G(u(t_0))$$

$$\frac{\rho^{\lambda}(t) u'(t)}{\phi(u(t))} \leq C_1 - \int_{t_0}^{t} \rho^{\lambda}(\tau) a(\tau) d\tau \quad (2.5)$$

Integrating on $[t_o, T]$ to obtain

$$\int_{t_o}^{T} \rho^{\lambda}(t) dG(u(t)) \leq C_1 (T - t_0) - \int_{t_0}^{T}\int_{t_0}^{t} \rho^{\lambda}(\tau) a(\tau) d\tau dt$$

$$\rho^{\lambda}(T) G(u(T)) - \rho^{\lambda}(t_0) G(u(t_0)) - \lambda \int_{t_0}^{T} G(u(t)) \rho^{\lambda-1}(t) \rho'(t) dt \leq$$

$$\leq C_1 (T - t_0) - \int_{t_0}^{T}\int_{t_0}^{t} \rho^{\lambda}(\tau) a(\tau) d\tau dt$$

Since $\rho > 0$, $\rho' \leq 0$ and $G(u(T)) \geq 0$, we obtain

$$\int_{t_0}^{T}\int_{t_0}^{t} \rho^{\lambda}(\tau) a(\tau) d\tau dt \leq C_2 + C_1 (T - t_0),$$

where $C_2 = \rho^{\lambda}(t_0) G(u(t_0))$.

Dividing by T, we have

$$\frac{1}{T}\int_{t_0}^{T}\int_{t_0}^{t} \rho^{\lambda}(\tau) a(\tau) d\tau dt \leq \frac{C_2}{T} + C_1\left(1 - \frac{t_0}{T}\right)$$

By taking limit sup and $T \to \infty$, from (2.3) we arrive at a contradiction. Similar result will be obtained when $u(t) < 0$, on $[t_o, \infty)$, for some $t_o \geq 0$.

Remark Condition (2.3) improves and extends the results in [2] and [5]. Condition (2.3) is another type oscillation for Eqn (1.1).

The following theorem weakens and extends the oscillation criterion in [1].

Theorem 2.3 Assume that Assumption A holds and there exists a function $\rho \in C^2[0,\infty]$ with $\rho > 0$, $\rho' \leq 0$ and $\rho'' \geq 0$ such that

$$\int_{\alpha}^{\infty} \rho^{\lambda}(t)a(t)dt = \infty, \qquad (2.6)$$

for some $\lambda \geq 1$ and $\alpha \geq 0$. Then Eqn (1.1) is oscillatory in $[0,\infty]$.

Proof Suppose that u(t) is a non oscillatory solution of Eqn (1.1) in $[0, \infty]$. Let $u(t) > 0$ on $[t_0,\infty)$ for some $t_0 \geq 0$. Let

$$v(t) = \frac{\rho^{\lambda}(t)u'(t)}{\phi(u(t))}, \; t \geq t_0$$

Arguing similar to proof of theorem 2.2, we obtain eqn (2.5)

$$\frac{\rho^{\lambda}(t)u'(t)}{\phi(u(t))} \leq C_1 - \int_{t_0}^{t} \rho^{\lambda}(\tau)a(\tau)d\tau, \qquad (2.7)$$

where $C_1 = v(t_0) - \lambda\rho^{\lambda-1}(t_0)\rho'(t_0)G(u(t_0))$.

From condition (2.6), it follows that there exists a $t_1 \geq t_0$ such that

$$\int_{t_0}^{t} \rho^{\lambda}(\tau)a(\tau)d\tau \geq 2|C_1|, \; t \geq t_1$$

Hence (2.7) becomes the inequality

$$\frac{\rho^{\lambda}(t)u'(t)}{\phi(u(t))} \leq -\frac{1}{2}\int_{t_0}^{t} \rho^{\lambda}(\tau)a(\tau)d\tau, \; t \geq t_1.$$

Integrating on $[t_1,T)$, we have

$$\rho^{\lambda}(T)G(u(T)) - \rho^{\lambda}(t_1)G(u(t_1)) - \lambda\int_{t_1}^{T} G(u(t))\rho^{\lambda-1}(t)\rho'(t)dt \leq$$

$$-\frac{1}{2}\int_{t_1}^{T}\int_{t_0}^{t} \rho^{\lambda}(\tau)a(\tau)d\tau dt$$

Since $\rho(T) > 0, G(u(T)) > 0, \rho'(t) \leq 0$ and $\lambda \geq 1$, we obtain

$$\int_{t_1}^{T}\int_{t_0}^{t} \rho^{\lambda}(\tau)a(\tau)d\tau dt \leq 2\rho^{\lambda}(t_1)G(u(t_1)) = K, \; T \geq t_1 \qquad (2.8)$$

From condition (2.6) we can choose a $t_2 \geq t_1$ such that

$$\int_{t_o}^{t} \rho^{\lambda}(\tau)a(\tau)d\tau \geq K,\ t \geq t_2$$

It follows that

$$\int_{t_1}^{T}\int_{t_o}^{t} \rho^{\lambda}(\tau)a(\tau)d\tau dt \geq \int_{t_2}^{T}\int_{t_o}^{t} \rho^{\lambda}(\tau)a(\tau)d\tau dt \geq K(T - t_2).$$

Letting $T \to \infty$ we have a contradiction to (2.8). Similar conclusion will be obtained when $u(t) < 0$ in $[t_o, \infty]$. Hence Eqn (1.1) is oscillatory in $[0, \infty]$.

3. CONCLUSIONS

Oscillation criteria have been proved by using assumption on f which is less weaker than used before, i.e the function $a(t)$ is not necessarily nonnegative on $[0, \infty)$.

The results in this paper will improve and extend the results in [2] and [5]. By using less weaker conditions in [1], we obtained another type of oscillation criteria for the equation.

The method in this paper can be used to establish oscillation criteria for nonlinear differential inequalities.

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