DESIGN PROCEDURE FOR OPTIMAL MULTI – INPUT CONTROL SYSTEMS VIA TIME – DOMAIN TECHNIQUE*

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ABSTRACT

Design procedures for Linear-optimal control systems with respect to a quadratic performance index are developed. Via a Time-domain Technique and based on the phase-variable canonical-system description, the optimal feedback vector $\tilde{\underline{k}}$ and the weighting matrix $\tilde{Q}$ can be directly determined from the characteristic equations of the open and closed loop systems. Formerly the design procedures has been developed for single input systems, and then to be extended to multi-systems.

SARI

PROSEDUR RANGCANGAN UNTUK SISTEM–SISTEM KONTROL OPTIMAL DENGAN MASUKAN BANYAK MELALUI BIDANG WAKTU

Suatu prosedur rancangan untuk Sistem-sistem Kontrol Optimal Linier dengan indeks performans kwadrat telah dikembangkan. Berdasarkan deskripsi sistem perubah pasa dan melalui bidang waktu, vektor catubalik optimal $\tilde{\underline{k}}$ dan matrik pembobotan $\tilde{Q}$ dapat ditentukan secara langsung dari persamaan-persamaan karakteristik sistem putaran terbuka dan putaran tertutup. Semula prosedur rancangan dikembangkan untuk sistem-sistem masukan tunggal dan kemudian dilanjutkan untuk sistem-sistem masukan banyak.

1. INTRODUCTION

Over the past few years a subtantial amount of literature has appeared dealing with the problem of optimal control systems with respect to a quadratic performance index. The general approach in solving of these problems is to choose an initial weighting matrices of the performance index, then to calculate the optimal feedback and the resulting system response is then obtained by simulation. Usually, the response is unsatisfactory, and so the performance index is modified by trial-and-error; the process is repeated until a saticfactory responce is achieved. These processes are made necessary to be done since the lack of knowledge of the relationship between the weighting matrices of the performance index and the response of the optimal system.

The purpose of this study is to develop a design procedure for the linear opti-

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mal control systems with respect to a quadratic performance index and simultaneously the closed-loop system performance can be achieved from a set of prescribed poles.

2. DESIGN PROCEDURE FOR SINGLE–INPUT SYSTEMS VIA TIME–DOMAIN TEHNIQUE

2.1 INTRODUCTION

The systems to be discussed are of the n^{th} order, linear, time-invariant, single-input, completely controllable and they are described by the vector-matrix equation:

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u \tag{2–1}$$

were $\underline{x}$ is the n state vector, u is the control input, A is a given $n \times n$ constant matrix and $\underline{b}$ is a given vector.
The optimization problem can be stated as follows:
First of all find linear feedback control law of the form:

$$u = -\underline{k}^T \underline{x} \tag{2–2}$$

so that the quadratic performance index

$$J = \int_0^\infty (\underline{x}^T Q \underline{x} + ru^2)\, dt \tag{2–3}$$

should be minimized, where Q is a symmetric non-negative-definite matrix, r is a positive constant scalar and superscript T denotes transpose.

It is well known that the optimal feedback vector $\underline{k}$ can be written as follows[1,5]

$$\underline{k}^T = \frac{1}{r}\underline{b}^T P \tag{2–4}$$

where P is the symmetric positive-definite matrix, which is a solution of the following Riccati equation:

$$PA + A^T P - \frac{1}{r} P\underline{b}\,\underline{b}^T P = -Q \tag{2–5}$$

If the weighting matrix Q and r are given, then the closed-loop poles are completely determined. In general, however, the resulting response of the close-lopp system may not be the desired one, but on the other hand, if the closed-loop poles are given, then the feedback vector $\underline{k}$ can be easily determined. However, the resulting system may not be an optimal system with respect to the requirement (2–3) that is minimized. In this work, a design procedure is developed and this procedure combines the two approaches to find $\underline{k}$ and matrix Q for a given r, so that the closed-loop system can achieve the desired poles and simultaneously the minimized quadratic performance index (2–3).

2.2 DEVELOPMENT OF DESIGN PROCEDURE

Without lost of generality, the systems to be treated in this chapter are written, in the following phase-variable-canonical form (2–6)[6]

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \text{ and } \underline{b} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \tag{2–6}$$

where $a_0, \ldots, a_{n-1}$ are the coefficients of the following open-loop characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$$

or

$$(\lambda - \lambda_1) \ldots (\lambda - \lambda_n) = 0 \tag{2–7}$$

where $\lambda_1, \ldots, \lambda_n$ are the open-loop poles.

Given $\lambda_1, \ldots, \lambda_n$ then $a_0, \ldots, a_{n-1}$ can be calculated or vice versa.

The closed-loop system equation can be described as follows:

$$\dot{\underline{x}} = (A - \frac{1}{r}\underline{b}\,\underline{b}^T P)\,\underline{x} \tag{2–8}$$

$$x = F\underline{x}$$

where the matrix F has the following canonical form:

$$F = A - \frac{1}{r}\underline{b}\,\underline{b}^T P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -f_0 & -f_1 & -f_2 & \cdots & -f_{n-1} \end{bmatrix} \tag{2–9}$$

and $f_0, \ldots, f_{n-1}$ are the coefficients of the following closed-loop characteristic equation

$$\lambda^n + f_{n-1}\lambda^{n-1} + \ldots + f_1\lambda + f_0 = 0$$

or

$$(\lambda - \alpha_1) \ldots (\lambda - \alpha_n) = 0 \tag{2–10}$$

where $\alpha_1, \ldots, \alpha_n$ are the desired closed-loop poles.

Given $\alpha_1, \ldots, \alpha_n$ ', then $f_0, \ldots, f_{n-1}$ can be calculated or vice versa.

Now, let us define vectors $\underline{a}$ and $\underline{f}$ as follows :

$$\underline{a}^T = (a_0, a_1, \ldots, a_{n-1}) \qquad (2-11)$$

and

$$\underline{f}^T = (f_0, f_1, \ldots, f_{n-1}) \qquad (2-12)$$

From eqns. (2–6), (2–9), (2–11) and (2–12), we obtain:

$$\underline{a} = -A^T \underline{b} \qquad (2-13)$$

and

$$\underline{f} = -F^T \underline{b} \qquad (2-14)$$

Since $P^T = P$ and $\underline{b}^T \underline{b} = 1$, from eqns. (2–9), (2–13) and (2–14), we could conclude that:

$$\begin{aligned} \underline{f} - \underline{a} &= (-F^T + A^T)\,\underline{b} \\ &= (\frac{1}{r}\underline{b}\,\underline{b}^T P)^T \underline{b} \qquad (2-15) \\ &= \frac{1}{r} P\underline{b} \end{aligned}$$

Hence, using eqns. (2–4) and (2–15), the optimal feedback vector is given by

$$\underline{k} = \underline{f} - \underline{a}$$

or

$$\underline{k}^T = (f_0 - a_0, \ldots, f_{n-1} - a_{n-1}) \qquad (2-16)$$

Eqn. (2–16) gives the optimal feedback vector $\underline{k}$ as an explicit function of the coefficients of the characteristic equations of the open and closed-loop system, and it forms the basis for the design procedure.

Using eqns. (2–15) and (2–16) and choosing $r = 1$ and Q as a diagonal matrix, the solution of eqn. (2–4) and the Riccati equation (2–5) can be uniquely determined in an interactive way by the following eqns. (2–17), (2–18) and (2–19), as follows.

$$p_{i,n} = f_{i-1} - a_{i-1}, \; i = 1, \ldots, n \qquad (2-17)$$

$$p_{i,j} = -p_{i-1,j+1} + (f_{i-1} f_j - a_{i-1} a_j) \qquad (2-18)$$

$$j \geqslant i$$

$$i, j = 1, \ldots, n-1$$

where

$$p_{o,i} = 0, \; i = 1, \ldots, n$$

and

$$q_{i,i} = -2p_{i-1,i+1} + (f^2_{i-1} - a^2_{i-1}) \qquad (2-19)$$

where $Q = \{q_{i,i}\}$ is a diagonal matrix and,

$P = \{p_{i,j}\}$ is a symmetric matrix.

If the desired closed-loop poles are located in the region of optimality, the resulting system will be an optimal one; that is to say P is a positive-definite matrix and Q is a non-negative-definite matrix[2]:

However, if the matrix Q is not chosesn as a diagonal matrix, there is no unique solution to eqns. (2–4) and (2–5) for P, Q and $\underline{k}$.
The optimal value of the performance index is given as

$$J^* = \underline{x}^T(0)\,P\underline{x}(0) \qquad (2\text{–}20)$$

where $\underline{x}(0)$ is a given initial state vector.
Now, one should revert to the original state variable description, using the following equations[6]:

$$\tilde{\underline{k}}^T = \underline{k}^T M^{-1} \qquad (2\text{–}21)$$

$$\tilde{P} = (M^{-1})^T PM^{-1} \qquad (2\text{–}22)$$

$$\tilde{Q} = (M^{-1})^T QM^{-1} \qquad (2\text{–}23)$$

$$\tilde{J}^* = J^* \qquad (2\text{–}24)$$

It should be noted that the restriction on Q as a diagonal matrix does not imply that $\tilde{Q}$ is a diagonal matrix. However, it can be shown that if Q is a non-negative-definite matrix then the same is valid for $\tilde{Q}$. The feedback vector $\tilde{\underline{k}}$ is considered to be an optimal one if the corresponding weighting matrix $\tilde{Q}$ is a non-negative-definite matrix.
The condition of optimality, can also be determined from the region of optimality of the closed-loop poles[2].
The procedure stated above will be summerized as follows:

1. Transform the original state variable discription into phase-variable canonical form,
2. Calculate the feedback vector $\underline{k}$ using eqn. (2–16),
3. Calculate the matrices Q and P from eqns. (2–17), (2–18) and (2–19),
4. Calculate the optimal value of the performance index from eqn. (2–20),
5. Then, one should revert to the original state variable description using eqns. (2–21), (2–22), (2–23) and (2–24).

Example will be discussed in chapter 3.

3. DESIGN PROCEDURE FOR OPTIMAL MULTI–INPUT SYSTEMS

3.1 I INTRODUCTION

In the previous chapter, the system to be treated are single-input systems. This is due to the computational difficulties that arose in handling polynomial matrices. The purpose of this chapter is to extend the design procedure presented in Chapter 2 to multi-input systems. Porter and Crossley[3], Fallside and Seraji[4] suggested an eqeivalency design technique for modal control system with multi-input. However, the procedure presented in this chapter is derived for modal and optimal control system.

3.2 DESIGN PROCEDURE

The procedure is mainly based on the equivalence of the closed-loop characteristic polynomial of a multi-input system and a corresponding single-input system.

Consider a multi-input system described by

$$\dot{\underline{x}} = A\underline{x} \quad B\underline{U} \tag{3–1}$$

and the quadratic performance index

$$J = \int_0^\infty (\underline{x}^T Q\underline{x} + \underline{U}^T R\underline{U})\, dt \tag{3–2}$$

is minimized, with the optimal control law

$$\underline{U}^* = -K^T \underline{x} \tag{3–3}$$

and the closed-loop system can achieve a set of prescribed closed-loop poles, where $\underline{x}$ is an n state vector, $\underline{U}$ is an m input vector, A is a constant n × n matrix, B is a constant n × m matrix, Q and R are, respectively, n × n non-negative-definite and m × m positive-definite matrices, K is an n × m feedback matrix.

The closed-loop poles are the roots of the following characteristic equation:

$$H(s) = |sI - A + BK^T| = 0 \tag{3–4}$$

Now, consider a single-input system described by

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u \tag{3–5}$$

and the performance index

$$J = \int_0^\infty (\underline{x}^T Q\underline{x} + ru^2)\, dt \tag{3–6}$$

is minimized, with the optimal control law

$$u^* = -\underline{k}^T \underline{x} \tag{3–7}$$

and the closed-loop system can achieve a set of prescribed closed-loop poles.

It should be noted that the vector $\underline{b}$ in eqn. (3–5) is not necessary of the

$$\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \quad \text{type.}$$

The closed-loop poles are the roots of the characteristic equation:

$$H(s) = |sI - A + \underline{b}\,\underline{k}^T| = 0 \tag{3–8}$$

If the roots of the eqns. (3–4) and (3–8) are required to be same value, then the following equations can be concluded[7].

$$K^T = \underline{d}\,\underline{k}^T \tag{3-9}$$

and $$B\underline{d} = \underline{b} \text{ for some m vector } \underline{d}. \tag{3-10}$$

Furthermore, the following equations can be obtained.

$$\underline{U} = \underline{d}u \tag{3-11}$$

and the weighting factor for the single-input system:

$$r = \underline{d}^T R\underline{d} \tag{3-12}$$

By making use of this equivalency technique, the design procedure for single-input systems presented in Chapter 2 can be extended to multi-input systems, once the vector $\underline{d}$ has been chosen.

3.3 THE CHOISE OF VECTOR $\underline{d}$

The choise of vector $\underline{d}$, in general, is arbritrary as long as the resulting equivalent single-input system is completely controllable or the open-loop poles are not cancelled in the open-loop transfer vector of the equivalent single-input system. The choice of vector $\underline{b}$ such that the equivalent single-input system is completely controllable, can be easily determined form the following open-loop transfer vector [1,5] : :

$$G(s) = (sI - A)^{-1}\,\underline{b} \tag{3-13}$$

From eqn. (3–10), however, the vector $\underline{d}$ can be uniquely determined once the vector $\underline{b}$ has been chosen, such that rank B = rank [B $\underline{b}$].

3.4 EXAMPLE

Consider the system

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \underline{U} \tag{3-14}$$

The open-loop poles are $\lambda_1 = -1$ and $\lambda_2 = -2$.

Find the optimal feedback matrix such that the closed-loop system achieves the poles at $\alpha_1 = -3$ and $\alpha_2 = -4$, and at the same time minimized the following quadratic performance index,

$$J = \int_0^\infty (\underline{x}^T Q\underline{x} + \underline{U}^T R\underline{U})\, dt$$

where $R = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ and Q is a n × n non-negative-definite matrix.

Let $\underline{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

Then, $\underline{b} = B\underline{d} = \begin{bmatrix} d_1 \\ d_2 + 2d_2 \end{bmatrix}$

Hence, the equivalent single-input system is found to be

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \underline{x} + \begin{bmatrix} d_1 \\ d_1 + 2d_2 \end{bmatrix} u \qquad (3-15)$$

The system of eqn. (3–15) will be a completely controllable system, if the vector $\underline{b}$ is to be chosen as :

$$\underline{b} = \begin{bmatrix} d_1 \\ d_1 + 2d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then, eqn. (3–10) yields

$$\underline{d} = \begin{bmatrix} 0 \\ 0{,}5 \end{bmatrix}$$

The system of eqn. (3–15) with the vector $\underline{d}$ given as above is already in phase-variable form. The problem now, can be solved by using procedure presented in Chapter 2 eqns. (2–11), (2–12) and (2–16). The vector $\underline{a}$ and $\underline{f}$ are found to be $\underline{a}^T = (2, 3)$ and $\underline{f}^T = (12, 7)$. Hence, eqn. (2–16) yields $\underline{k}^T = (10, 4)$ Now, one should revert to the multi-input (original) system using eqn. (3–9) and we get the optimal feedback matrix as follows:

$$K^T = \begin{bmatrix} 0 & 0 \\ 5 & 2 \end{bmatrix}$$

Using eqn. (3–12) we obtain that r = 1. Hence the matrices P and Q can be determined by using eqns. (2–17), (2–19) in chapter 2. By choosing Q as a diagonal matrix, we find that:

$$P = \begin{bmatrix} 78 & 10 \\ 10 & 4 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 140 & 0 \\ 0 & 20 \end{bmatrix}$$

It should be noted that the multi-input system and the equivalent single-input system should have the same matrix P and matrix Q.

4. CONCLUDING REMARKS

The objective of this study is to develop a design procedure for a linear optimal control system with Prescribed closed-loop poles. Via a time-domain technique and based on the phase-variable canonical-system description, the optimal feedback vector $\tilde{\underline{k}}$ and the weighting matrix $\tilde{Q}$ can be directly determined from the caracteristic equations of the open and open-loop system. This design procedure is general, that can be applied to any order of the system and the type poles, i.e., real, complex, distinct or multiple.

The procedure developed for single-input systems has been extended to multi-input systems based on the equivalence of the closed-loop characteristic polynomials of a multi-input system and a corresponding single-input system.

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