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OSCILLATION THEOREMS FOR SECOND ORDER DELAY EQUATIONS

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R I N G K A S A N

Dalam tulisan ini kriteria oksilasi untuk suatu persamaan diferensial, diperluas.

Hasil yang diperoleh dari perluasan itu menjadi berbentuk

$$(r(t)y'(t))' + F(t,y(t),y(g(t))) = 0$$

untuk $t \geq a > 0$.

A B S T R A C T

Recently, oscillation criteria for certain second order delay differential equations have been substantially studied by [2], [3], [5], *and others.*

In this paper, we generalize and extend some of their results to a more general delay differential equation of the type

$$(r(t)y'(t))' + F(t,y(t),y(g(t))) = 0$$

$$t \geq a > 0.$$

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1. INTRODUCTION

Consider the following delay differential equation

$$(r(t)y'(t))' + F(t,y(t),y(g(t))) = 0 \quad t \geq a > 0 \qquad (1\text{-}1)$$

where $F \in C([a,\infty) \times R \times R, R)$; $g \in C([a,\infty),R)$; $r \in C'([a,\infty)R)$; $r(t) > 0$ for $t \geq a$; and $g(t) \to \infty$ as $t \to \infty$.

Clearly, Eq.(1-1) is more general than the following two differential equations

$$(r(t)y'(t))' + p(t)\ f(y(t),y(g(t))) = 0 \quad t \geq a > 0 \qquad (1\text{-}2)$$

where $f \in C(R \times R,R)$; p, g are assumed continuous on $[a,\infty)$; $r \in C'([a,\infty),R)$; $r(t) > 0$ and $p(t) \geq 0$ for $t \geq a$; and $g(t) \to \infty$ as $t \to \infty$.

$$y''(t) + F(t,y(t),y(g(t))) = 0 \quad t \geq a > 0 \qquad (1\text{-}3)$$

where $F \in C([a,\infty) \times R \times R,R)$; $g \in C([a,\infty), R)$, and $g(t) \to \infty$ as $t \to \infty$.

Recently, Chiou [3], Erbe [5], and Ladas [8] have obtained some oscillation criteria for Eq.(1-3) and Bradley [2] for Eq.(1-2). The purpose of this paper is to present conditions for the oscillatory of solutions of Eq.(1-1). We would see that oscillation criteria depend significantly on the rate of increase of the function $g(t)$ and the boundedness of the function $r(t)$. The main results of the paper generalize and extend some of the previous results of Bradley [2], Chiou [3], Kung [7], Travis [10], Waltman [11], and Wong [13]. These results are obtained by using integral inequality and a well-known theorem of Wintner [12]. As a consequence of our main results, an oscillation criteria is given for Eq.(1-2).

Consider the equation

$$y''(t) + \lambda p(t)\ y(t) = 0 \qquad (1\text{-}4)$$

We shall call $p(t)$ a strongly oscillatory coefficient if (1-4) is oscillatory for all positive λ. For the case $p(t) \geq 0$, Nehari [9] has shown that

$$\limsup_{t\to\infty} t \int_t^{\infty} p(s)\,ds = \infty$$

is a necessary and sufficient condition for $p(t)$ to be a strongly oscillatory coefficient.

A nontrivial solution to (1-1), (1-2) or (1-3) is called oscillatory if it has arbitrarily large zeroes in $[a,\infty)$, and nonoscillatory if it is eventually of one sign on $[a,\infty)$.

Eqs.(1-1), (1-2) or (1-3) is said to be oscillatory if its every solution is oscillatory. We shall assume the existence of nontrivial solutions of (1-1), (1-2) or (1-3) in the interval $[a,\infty)$. A general discussion of existence and uniqueness of solutions to delay equations is given in El'sgol'ts [4].

2. *OSCILLATION CRITERIA*

In order to prove the oscillation theorem, we need the following Lemma. Its proof is similar to those of Bradley [2, Lemma 2] and Erbe [6, Lemma 1-2].

Lemma 1. Consider the differential equation (1-1). Suppose that the following conditions are satisfied:

(i) $F(t,u,v)$ has the sign of u and v when they have the same sign for $t \geq a$;

(ii) $F(t,u,v)$ is nondecreasing in both u and v for $t \geq a$; and

(iii) $\int^{\infty} \frac{dt}{r(t)} = \infty.$

Then, if $y(t)$ is a solution of (1-1) that is eventually positive (negative), $y'(t) > 0$ ($y'(t) < 0$) for all large t.

Proof. Let $y(t) > 0$ for all large t. Firstly, we show that $y'(t) \geq 0$ for all large t. If this is not true, then there exists a t_0 larger than the last zeroes of $y(t)$ and $y(g(t))$ such that $y'(t_0) < 0$. For $t \geq t_0$

$$(r(t)y'(t))' = -F(t,y(t),y(g(t))) \leq 0 \qquad (2\text{-}1$$

Integrating (2-1) from t_o to t, we have

$$r(t)y'(t) \leq r(t_o)y'(t_o) < 0$$

Hence, $$y'(t) \leq \frac{r(t_o)y'(t_o)}{r(t)} \quad \text{for } t \leq t_o. \tag{2-2}$$

Integrating (2-2) from t_o to t, and letting $t \to \infty$, we see that $y(t)$ is eventually negative. This contradicts the hypothesis.

Next, let us assume that $\{T_n\}$ is a sequence of real numbers at which $y'(T_n) = 0$, $\forall\, n$, and $T_n \to \infty$. Thus, it follows from the above result that $y(t)$ is nondecreasing for t sufficiently large. Therefore it has a limit (finite positive or $+\infty$). This implies that there exist a constant $c > 0$ and a sufficiently large t_1 such that $y(t) > c$ and $y(g(t)) > c$ for all $t \geq t_1$. By conditions (i) and (ii), $F(t,y(t),y(g(t))) \geq F(t,c,c) > 0$ for $t \geq t_1$. Thus, by choosing a $T_k \in \{T_n\}$ so that $T_k \geq t_1$, we obtain from Eq.(1-1) that for all $t > t_k$

$$(r(t)y'(t))' \leq - F(t,c,c) \tag{2-3}$$

Integrating (2-3) from T_n to T_{n+1} $(n \geq k)$, summing over the indices n and using the fact that $F(t,c,c) > 0$ for $t \geq T_k$, we obtain a contradiction. Thus, we have that $y'(t) > 0$ or $y'(t) \equiv 0$ for all large t. However, if $y(t) \equiv 0$ for all large t, then $y(t) = c$ for some constant $c > 0$. A contradiction follows immediately from Eq.(1-1) and condition (i). Therefore, $y'(t) > 0$ for all large t. The proof remains the same if $y(t) < 0$ for all large t. This completes the proof.

With the help of Lemma 1, we prove the following tehorem.

Theorem 1: Consider the differential equation (1-1). Suppose that the following conditions are satisfied

(i) $F(t,u,v)$ has the sign of u and v when they have the same sign for $t \geq a$;

(ii) $F(t,u,v)$ is nondecreasing in both u and v for $t \geq a$;

(iii) $\int^{\infty} \frac{dt}{r(t)} = \infty$; and

(iv) $\int^{\infty} |F(t,c,c)|\,dt = \infty$, for every constant $c \neq 0$.

Then Eq.(1-1) is oscillatory.

Proof: Let $y(t)$ be a nonoscillatory solution of (1-1). Since all the assumptions of Lemma 1 are satisfied, it follows from Lemma 1, that we may assume $y(t) > 0$ and $y'(t) > 0$ for all $t \geq t_o \geq a$. This implies that $y(t) \to L$ as $t \to \infty$, where L is a finite positive or $+\infty$. In either case, there exist a constant $c > 0$ and a t_1, $t_1 \geq t_o$, such that $y(t) > c$ and $y(g(t)) > c$ for all $t \geq t_1$.

By conditions (i) and (ii), it follows from integrating (1-1) from t_1 to t that for all $t \geq t_1$

$$r(t)y'(t) = r(t_1)y'(t_1) - \int_{t_1}^{t} F(s,y(s),y(g(s)))\,ds$$

$$\leq r(t_1)y'(t_1) - \int_{t_1}^{t} F(s,c,c)\,ds \qquad (2\text{-}4)$$

Thus, we obtain from condition (iv) that $r(t)y'(t)$ is eventually negative for sufficiently large t. Further, since $r(t) > 0$, it is clear that $y'(t)$ is eventually negative. This contradicts the fact that $y'(t) > 0$ for all $t \geq t_o$. A similar argument applies to the case in which $y(t) < 0$ for all large t. This completes the proof.

Note that Theorem 1 contains Theorem 3-1 of Ladas [8] in the case $n = 2$ as a special case.

Remark 1. The condition (iv) of Theorem 1 is not a necessary condition as shown in the following example.

Consider the equation

$$y''(t) + \frac{1}{2t^2}\,y(t) = 0$$

Clearly, the condition (iv) is not satisfied. However, the equation is oscillatory, (see Bellman [1]. Theorem 10, p. 121).

In order to obtain the desired oscillation criteria for Eq.(1-1), we shall impose some restrictions on g(t) and r(t). Suppose that there exists a differentiable fucntion q(t) such that

$$q(t) \leq \min \{g(t), t\} \text{ and } q'(t) \geq \gamma > 0 \qquad (3\text{-}1)$$

for sufficiently large t.

We prove the folowing theorem using a well-known theorem of Wintner [12, part 4, p.371].

Theorem 2. Consider the differential equation (1-1). Let g(t) and q(t) satisfy (3-1) and suppose that

(i) F(t,u,v) has the sign of u and v when they have the same sign for $t \geq a$;

(ii) F(t,u,v) is nondecreasing in both u and v for $t \geq a$;

(iii) there exists a positive nondecreasing continuous function k(t) for $t \geq a$, and a constant $M > 0$ such that $|u| > M$ implies

$$\liminf_{|v|\to\infty} \left|\frac{k(\alpha|v|)F(t,u,v)}{v}\right| \geq \varepsilon\, |F(t,c,c)| \qquad (3\text{-}2)$$

for every constant $\alpha > 0$, for some constant $c \neq 0$, for some $\varepsilon > 0$ and for sufficiently large t.

(iv) $\frac{1}{r(t)} \geq \lambda > 0$ and $r'(t) \geq 0$ for sufficiently large t; and

(v) $$\limsup_{t\to\infty} t \int_t^\infty \frac{|F(s,c,c)|}{k(g(s))}\, ds = \infty \qquad (3\text{-}3)$$

for every constant $c \neq 0$. Then Eq.(1-1) is oscillatory.

Proof: Let y(t) be a nonoscillatory solution of Eq.(1-1). Then, by (iv) and Lemma 1, we may assume that $y(t) > 0$ and $y'(t) > 0$ for all large $t \geq t_o \geq a$. (The case $y(t) < 0$ can be similarly treated). Choose a $t_1 \geq t_o$ so that $q(t) \geq t_o$ and $r'(t) \geq 0$ for $t \geq t_1$. Then it is easily verified from Eq.(1-1) that $y''(t) \leq 0$ for $t \geq t_1$.

Define $$w(t) = \frac{r(t)y'(t)}{y(q(t))} \quad \text{for } t \geq t_1.$$

Clearly, for $t \geq t_1$, $w(t)$ satisfies

$$w'(t) + \frac{F(t,y(t),y(g(t)))}{y(q(t))} + \frac{w(t)y'(q(t))q'(t)}{y(q(t))} = 0 \qquad (3\text{-}4)$$

Since $y(t)$ is increasing and positive for all $t \geq t_o$, it has a limit (a finite positive or $+\infty$). We shall discuss these two cases separately.

(i) If $\lim_{t\to\infty} y(t) = L$, where $0 < L < \infty$, then we may choose a $t_2 > t_1$, so that, for $t \geq t_2$

$$y(t) > \tfrac{1}{2} L \text{ and } y(g(t)) > \tfrac{1}{2} L \qquad (3\text{-}5)$$

Since $y'(t)$ is nonincreasing for $t \geq t_1$, it follows from the mean value theorem that $0 < \alpha\, y(t) \leq t$ for some positive constant α and for all large $t \geq t_3 \geq t_2$. Thus, if we choose a $t_4 \geq t_3$ so that $g(t) \geq t_3$ for $t \geq t_4$, we have

$$\alpha\, y(g(t)) \leq g(t) \qquad (3\text{-}6)$$

for $t \geq t_4$.

Since F,k are nondecreasing, and $y(t)$ is increasing for $t \geq t_2$, it follows from (3-1), (3-5) and (3-6) that for $t \geq t_4$

$$\frac{F(t,y(t),y(g(t)))}{y(q(t))} \geq \frac{k(\alpha y(g(t)))\ F(t,y(t),y(g(t)))}{k(\alpha y(g(t)))\cdot y(t)}$$

$$\geq \frac{k(\tfrac{1}{2}\alpha L)\ F(t,\tfrac{1}{2}L,\tfrac{1}{2}L)}{k(g(t))\cdot L}$$

$$= \beta\, \frac{F(t,\tfrac{1}{2}L,\tfrac{1}{2}L)}{k(g(t))} \qquad (3\text{-}7)$$

where $= \frac{k(\tfrac{1}{2}\alpha L)}{L} > 0$.

(ii) Now, suppose that $y(t) \to \infty$ as $t \to \infty$. Since $y'(t)$ is nonincreasing for $t \geq t_1$, it follows from the mean value theorem that $0 < \alpha\, y(t) \leq t$ for some positive constant α and for all large $t \geq t_5 \geq t_1$. Thus, if we choose a $t_6 \geq t_5$ so that $g(t) \geq t_5$ for $t \geq t_6$, we have

$$0 < \alpha\, y(g(t)) \leq g(t) \tag{3-8}$$

for $t \geq t_6$.

Since F,k are nondecreasing, and y(t) is increasing, we deduce from (3-1), (3-8) and condition (iii), that for $t \geq t_7 \geq t_6$

$$\begin{aligned} \frac{F(t,y(t),y(g(t))}{y(q(t))} &\geq \frac{k(\alpha y(g(t)))\ F(t,y(t),y(g(t)))}{k(\alpha(y(g(t)))\ y(g(t))} \\ &\geq \frac{k(\alpha y(g(t)))\ F(t,y(t),y(g(t)))}{k(g(t))\ y(g(t))} \\ &= \varepsilon \frac{F(t,c,c)}{k(g(t))} \end{aligned} \tag{3-9}$$

for some $\varepsilon > 0$, and for some constant $c \neq 0$.

In view of inequality (3-7) and inequality (3-9), we see that (3-9) holds true for both cases, for $t \geq t_7$. Further, since $y'(t)$ is nonincreasing for $t \geq t_1$, we have

$$\begin{aligned} \frac{w(t)\ y'(q(t))\ q'(t)}{y(q(t))} &\geq \gamma\ w(t)\ \frac{y'(t)r(t)}{y(q(t))r(t)} \\ &\geq \gamma\ \lambda\ w^2(t) \end{aligned} \tag{3-10}$$

Thus, (3-4) is reduced to

$$w'(t) + \varepsilon \frac{F(t,c,c)}{k(g(t))} + \gamma\ \lambda\ w^2(t) \leq 0 \tag{3-11}$$

Now, let $R(t) = \gamma\ \lambda\ w(t)$. Then, we obtain the following Riccati inequality from (3-11)

$$R'(t) + R^2(t) + \gamma\lambda\varepsilon \frac{F(t,c,c)}{k(g(t))} \leq 0 \qquad (3\text{-}12)$$

Thus, it follows from the result of Wintner [12, part 4, p. 371], that the equation

$$y''(t) + \gamma\lambda\varepsilon \frac{F(t,c,c)}{k(g(t))} y(t) = 0 \qquad (3\text{-}13)$$

is nonoscillatory.

But this contradicts the fact that condition (v) is a necessary and sufficient condition for $\frac{F(t,c,c)}{k(g(t))}$ to be a strongly oscillatory coefficient for Eq.(3-13). Thus completes the proof.

Remark 2. If $g(t) \leq t$ in Theorem 2, the conditions (iii) and (iv) can be replaced, respectively, by

(iii)' there exists a positive nondecreasing continuous function $k(t)$ for $t \geq a$ such that for sufficiently large t

$$\liminf_{\substack{|v|\to\infty \\ |u|\geq|v|}} \left|\frac{k(\alpha|v|)F(t,u,v)}{v}\right| \geq \varepsilon\ |F(t,c,c)| \qquad (3\text{-}14)$$

for every constant $\alpha > 0$, for some $\varepsilon > 0$, and for some constant $c \neq 0$

(iv)' $$\limsup_{t\to\infty} t \int^{\infty} \frac{|F(s,c,c)|}{k(s)}\, ds = \infty \qquad (3\text{-}15)$$

for every constant $c \neq 0$.

Consider the following equations

$$y''(t) + \tfrac{1}{4}(t^3 \ell n\ t)^{-\frac{1}{2}}\, y(\ell n\ t) = 0 \quad t > 1 \qquad (3\text{-}16)$$

and

$$(t^{\frac{1}{4}}\, y'(t))' + \tfrac{1}{4}\, t^{-7/4}\, y(\tfrac{1}{4}\, t) = 0 \quad t > 1 \qquad (3\text{-}17)$$

We see that both equations have nonoscillatory solutions $y(t) = \sqrt{t}$. However, in view of Eq.(1-4) we note that $p(t) = \frac{1}{4}(t^3 \ln t)^{-1/2}$ and $p(t) = \frac{1}{4} t^{-7/4}$ are strongly oscillatory coefficients. Thus, the assumptions on the rate of increase of $g(t)$ and the boundedness of $r(t)$ given in Theorem 2 are necessary.

We shall prove the following theorem.

Theorem 3: Consider the differential equation (1-1). Let $g(t)$ and $q(t)$ satisfy (3-1), and suppose that conditions (i), (ii), (iii) and (iv) of Theorem 2 are satisfied. Then, if

$$\int^{\infty} \left\{\lambda\gamma \frac{t|F(t,c,c)|}{k(g(t))} - \frac{1}{4t}\right\} dt = \infty \tag{4-1}$$

for every constant $c \neq 0$, Eq.(1-1) is oscillatory.

Proof: Let $y(t)$ be a nonoscillatory solution of (1-1). Using the argument similar to that of Theorem 2, we may assume that $y(t) > 0$, $y(q(t)) > 0$, $y'(t) > 0$, and $y''(t) \leq 0$ for all $t \geq t_1 \geq a$.

Define $$w(t) = -\frac{t\, r(t) y'(t)}{y(q(t))} \quad \text{for } t \geq t_1.$$

Differentiating the expression for $w(t)$, it follows from Eq.(1-1) that for $t \geq t_1$

$$w'(t) = \frac{t\, F(t,y(t),y(g(t)))}{y(q(t))} - \frac{r(t)y'(t)}{y(q(t))} + \frac{t\, r(t)y'(t)(y(q(t)))'}{y^2(q(t))} \tag{4-2}$$

Arguing as for theorem 2, we have for all $t \geq T_o \geq t_1$

$$\frac{F(t,y(t),y(g(t)))}{y(q(t))} \geq \varepsilon \frac{F(t,c,c)}{k(g(t))} > 0 \tag{4-3}$$

for some constant $c \neq 0$, and some ε which is chosen in such a way that $0 < \varepsilon < 1$. Further, since $y'(t)$ is nondecreasing for $t \geq t_1$, it follows from the argument similar to that of Theorem 2 that

$$\frac{t\ r(t) y'(t) (y(q(t)))'}{y^2(q(t))} \geq \lambda\ \gamma\ \frac{w^2(t)}{t} \tag{4-4}$$

By (4-3) and (4-4), we see that Eq.(4-2) can be reduced to

$$w'(t) \geq \varepsilon\ \frac{t\ F(t,c,c)}{k(g(t))} + \frac{w(t)}{t} + \lambda\ \gamma\ \frac{w^2(t)}{t} \tag{4-5}$$

for $t \geq T_o$.

Since $0 < \varepsilon < 1$, Eq.(4-5) becomes

$$\frac{1}{\varepsilon}\, w'(t) \geq \frac{t\ F(t,c,c)}{k(g(t))} + \frac{w(t)}{t} + \lambda\ \gamma\ \frac{w^2(t)}{t}$$

Let $H(t) = \sqrt{\lambda\gamma}\ w(t) + \frac{1}{2\sqrt{\lambda\gamma}}$. Then, for $t \geq T_o$

$$\frac{w'(t)}{\varepsilon} = \frac{H'(t)}{\varepsilon\sqrt{\lambda\gamma}} \geq \frac{t\ F(t,c,c)}{k(g(t))} + \frac{H^2(t)}{t} - \frac{1}{4\lambda\gamma\ t} \tag{4-6}$$

Integrating (4-6) from T_o to t, we obtain

$$\frac{H(t)}{\varepsilon\sqrt{\lambda\gamma}} \geq \frac{H(T_o)}{\varepsilon\sqrt{\lambda\gamma}} + \int_{T_o}^{t} \left(\frac{s\ F(s,c,c)}{k(g(s))} - \frac{1}{4\lambda\gamma s}\right) ds + \int_{T_o}^{t} \frac{H^2(s)}{s}\, ds \tag{4-7}$$

By (4-1) and the fact that $\lambda\gamma > 0$, we see that the sum of the first two terms on the right side of (4-7) will be positive for sufficiently large $t \geq T_1 \geq T_o$, and hence for $t \geq T_1$

$$\frac{H(t)}{\varepsilon\sqrt{\lambda\gamma}} \geq \int_{T_o}^{t} \frac{H^2(s)}{s}\, ds = K(t)$$

Thus, $$K'(t) = \frac{H^2(t)}{t} \geq \varepsilon^2\lambda\gamma \frac{K^2(t)}{t}\ , \text{ and}$$

$$\frac{1}{t} \leq \frac{1}{\varepsilon^2\lambda\gamma} \frac{K'(t)}{K^2(t)} \quad \text{for } t \geq T_1 \tag{4-8}$$

Integrating (4-8) from T_1 to t, and letting $t \to \infty$, we obtain

$$\int_{T_1}^{\infty} \frac{1}{t}\, dt \leq \frac{1}{\varepsilon^2\lambda\gamma\ K(T_1)} < \infty$$

This is a contradiction. The case $y(t) < 0$ can be treated in the same way. This completes the proof.

Note that we can see from Theorem 3 that the assumptions on the rate of increase of $g(t)$ and the boundedness of $r(t)$ are required for (4-1) to hold.

Remark 3

(i) In particular, Theorem 3 shows that Eq.(1-1) is oscillatory, if

$$\liminf_{t\to\infty} \frac{t^2|F(t,c,c)|}{k(g(t))} > \frac{1}{4\lambda\gamma} \tag{4-9}$$

for every constant $c \neq 0$.

(ii) If $g(t) \leq t$ in Theorem 3, condition (iii) can be replaced by (iii)' of Remark 2, and (4-1) by

$$\int^{\infty} \left\{\lambda\gamma \frac{t|F(t,c,c)|}{k(t)} - \frac{1}{4t}\right\} dt = \infty \tag{4-10}$$

for every constant $c \neq 0$.

As a consequence of Theorem 3, we have the following corollaries.

Corollary 1: Consider the differential equation (1-1). Let $g(t)$ and $q(t)$ satisfy condition (3-1), and suppose that conditions (i), (ii), (iii), and (iv) or Theorem 2 are satisfied. Further, if

$$\int^{\infty} \frac{t^{\delta}|F(t,c,c)|}{k(g(t))}\,dt = \infty \tag{4-11}$$

for every constant $c \neq 0$, and for some $\delta < 1$. Then, Eq.(1-1) is oscillatory.

Proof: From (4-11) it is clear that for any $\varepsilon > 0$

$$\frac{t^{\delta}|F(t,c,c)|}{k(g(t))} \geq \frac{1}{t^{1+\varepsilon}} \tag{4-12}$$

for all $t \geq T_o \geq a$. Thus, if ε is chosen so that $\varepsilon + \delta < 1$, we have

$$\left(\lambda\gamma \frac{t\ |F(t,c,c)|}{k(g(t))} - \frac{1}{4t}\right) \geq \frac{\gamma\lambda}{t^{\varepsilon+\delta}} - \frac{1}{4t} \geq \frac{1}{t}\left(\lambda\gamma\ t^{1-(\varepsilon+\delta)} - \frac{1}{4}\right)$$

Therefore, for all $t \geq T_1 \geq T_o$

$$\left(\lambda\gamma \frac{t\ |F(t,c,c)|}{k(g(t))} - \frac{1}{4t}\right) \geq \frac{1}{t} \tag{4-13}$$

Integrating (4-13) from T_1 to t, and letting $t \to \infty$, we obtain (4-1). Thus, it follows from Theorem 3 that Eq.(1-1) is oscillatory.

In view of the inequality (4-12), it is readily seen that the condition (4-11) implies the condition (3-3).

Using the similar technique as for Corollary 1, we have

Corollary 2: Consider the differential equation (1-1). Let $g(t)$ and $q(t)$ satisfy condition (3-1), and suppose that conditions (i), (ii), (iii) and (iv) of Theorem 2 are satisfied. Further, if

$$\int^{\infty} \frac{t\,|F(t,c,c)|}{(\log g(t))^{\beta}\,k(g(t))}\,dt = \infty \tag{4-14}$$

for every constant $c \neq 0$, and for some $\beta > 1$. Then Eq.(1-1) is oscillatory.

Proof: Clearly, the condition (4-14) implies that for any given $\alpha > 1$

$$\frac{t\,|F(t,c,c)|}{(\log g(t))^{\beta}\,k(g(t))} \geq \frac{1}{t(\log t)^{\alpha}} \quad \text{for all } t \geq t_o \geq a$$

Thus, choosing $\alpha < \beta$, and using the fact that $q(t) \geq b\,t$ for some b with $0 < b \leq \gamma$ and for $t \geq t_1 \geq a$, we have

$$\gamma\lambda\,\frac{t\,|F(t,c,c)|}{k(g(t))} \geq \lambda\gamma\,\frac{(\log g(t))^{\beta}}{t(\log t)^{\alpha}} \geq \lambda\gamma\,\frac{(\log q(t))^{\beta}}{t(\log t)^{\alpha}}$$

$$\geq \lambda\gamma\,\frac{(\log bt)^{\beta}}{t(\log t)^{\alpha}} \geq \frac{\lambda\gamma}{t}\left(\frac{\log bt}{\log t}\right)^{\alpha}(\log bt)^{\beta-\alpha} \tag{4-15}$$

for $t \geq t_2 = \max\,\{t_o, t_1\}$.

Since $\frac{\log bt}{\log t} \to 1$ as $t \to \infty$, there exists for every given ε with $0 < \varepsilon < 1$, a $t_3 \geq a$ such that $\frac{\log bt}{\log t} > 1 - \varepsilon$ for $t \geq t_3$. Thus, for $t \geq t_4 = \max\,\{t_2, t_3\}$, the inequality (4-15) becomes

$$\lambda\gamma\,\frac{t\,|F(t,c,c)|}{k(g(t))} \geq \lambda\gamma\,\frac{(1-\varepsilon)^{\alpha}\,(\log bt)^{\beta-\alpha}}{t}$$

Therefore,

$$\left(\lambda\gamma \frac{t\,|F(t,c,c)|}{k(g(t))} - \frac{1}{4t}\right) \geq \frac{1}{t}\left(\lambda\gamma\ (1-\varepsilon)^{\alpha}\ (\log bt)^{\beta-\alpha} - \frac{1}{4}\right)$$

$$\geq \frac{1}{t} \text{ for all } t \geq t_5 \geq t_4 \qquad (4\text{-}16)$$

Integrating (4-16) from t_5 to t, and letting $t \to \infty$, we obtain (4-1). Thus, by Theorem 3, Eq.(1-1) is oscillatory.

Applying Theorem 2 and Theorem 3 to the Eq.(1-2) we obtain the following Corollary.

Corollary 3: Consider the differential equation (1-2). Let g(t) and q(t) satisfy the condition (3-1), and suppose that the following conditions are satisfied

(i) f(u,v) has the sign of u and v when they have the same sign;

(ii) f(u,v) is nondecreasing in both u and v;

(iii) there exists a positive nondecreasing continuous function k(t) for $t \geq a$, and a constant $M > 0$ such that $|u| > M$ implies

$$\liminf_{|v|\to\infty} \left|\frac{k(\alpha|v|)\ f(u,v)}{v}\right| \geq \varepsilon > 0 \qquad (5\text{-}1)$$

for every constant $\alpha > 0$ and for some ε.

(iv) $\frac{1}{r(t)} \geq \lambda > 0$, and $r'(t) \geq 0$ for all large t;

(v) either $$\limsup_{t\to\infty} t \int_t^{\infty} \frac{p(s)}{k(g(s))}\, ds = \infty \qquad (5\text{-}2)$$

or

$$\int^{\infty} \left(\lambda\gamma \frac{tp(t)}{k(g(t))} - \frac{1}{4t}\right) dt = \infty. \qquad (5\text{-}3)$$

Then, Eq.(1-2) is oscillatory.

Remark 4

(1) If $g(t) \leq t$ in Corollary 3, conditions (iii) and (iv) can be replaced, respectively, by

(iii)' there exists a positive nondecreasing continuous function $k(t)$ for $t \geq a$ such that

$$\lim_{\substack{|v|\to\infty \\ |u|\geq|v|}} \inf \left|\frac{k(\alpha|v|)\ f(u,v)}{v}\right| > 0 \tag{5-4}$$

for every constant $\alpha > 0$,

(iv)' either
$$\limsup_{t\to\infty} t \int_t^\infty \frac{p(s)}{k(s)}\, ds = \infty \tag{5-5}$$

or

$$\int^\infty \left\{\lambda\gamma \frac{tp(t)}{k(t)} - \frac{1}{4t}\right\} dt = \infty \tag{5-6}$$

(2) In the case $g(t)$ and $r(t)$ satisfy (3-1) and condition (iv) of Theorem 2, respectively, Corollary 3 contains Theorem 2 of Bradley [2] as a special case. In particular, if $r(t) = 1$, Corollary 3 contains the results of Waltman [11], and Theorem 2-2 of Travis [10], as special cases.

Consider the following equation

$$y''(t) + p(t)y(g(t)) = 0 \quad t \geq a \geq 0 \tag{5-7}$$

where p and g are continuous on $[a,\infty]$; $p(t) \geq 0$, and $g(t) \to \infty$ as $t \to \infty$. Eq.(5-7) is a special case of Eq.(1-2) when $r(t) = 1$, and $f(u,v) = v$. If $k(t) = 1$ for $t \geq a$, we have the following corollary.

Corollary 4: Consider the differential equation (5-7). Let $g(t)$ and $q(t)$ satisfy (3-1). Then, if either one of the following two conditions are satisfied

$$\text{(a)} \quad \limsup_{t\to\infty} t \int_t^\infty p(s)\,ds = \infty \tag{5-8}$$

$$\text{(b)} \quad \int^\infty \left\{\gamma t\ p(t) - \frac{1}{4t}\right\} dt = \infty, \tag{5-9}$$

Eq.(5-7) is oscillatory.

Remark 5. Corollary 4 improves Theorem 4-1 of Wong [13], Corollary 2-4 of Erbe [5], Corollary 4-1 of Kung [7], and Corollary 2-4 of Chiou [3].

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