PROCEEDINGS ITB Vol. 9, No. 3, 1975.

THE MULTIPLICATION PROBLEM FOR SPHERES

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R I N G K A S A N

Masalah pendarapan bagi bola adalah masalah untuk menentukan bola-bola dalam Ruang Euclid $S^{n-1} \to E^n$ membenarkan suatu pendarapan yang kontinu. Tulisan ini menyampaikan bukti teori K topologi bahwa hanya dapat terjadi apabila $n = 1, 2, 4,$ dan 8. Kasus ini bersesuaian dengan $S^0 \to E^1$, $S^1 \to E^2$, $S^3 \to E^4$, dan $S^7 \to E^8$ di mana pendarapan-pendarapan diberikan oleh bilangan-bilangan riil, kompleks, kuaternion, dan bilangan Cayley.

A B S T R A C T

The multiplication problem for spheres is to determine which spheres in Euclidean space $S^{n-1} \to E^n$ permit a continuous multiplication. This paper presents the topological K-theory proof that it is only possible when $n = 1, 2, 4,$ and 8. These cases correspond to $S^0 \to E^1$, $S^1 \to E^2$, $S^3 \to E^4$, and $S^7 \to E^8$ where the multiplications are given respectively by the real numbers, complex numbers, quaternions, and Cayley numbers.

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In this paper we present a proof of the multiplication problem for spheres, one which illustrates the power of K-theory. A *multiplication with unit* on an (n-1) sphere S^{n-1} is a map (maps will always be continuous)

$$m : S^{n-1} \times S^{n-1} \to S^{n-1}$$

satisfying the additional property

$$m(e,x) = m(x,e) = x, \; x \in S^{n-1}$$

where e is some fixed base point. The multiplication problem for spheres is simply to determine for what n such a multiplication m exists. It does exist for $n = 1, 2, 4$, and 8. The case $n = 1$ is trivial, and, for the cases $n = 2, 4$, and 8, multiplications are given respectively by the complex numbers, quaternions, and Cayley numbers. These are in fact the only n. This answer was only obtained around 1960 by J.F. Adams after at least 25 years of investigation by many leading mathematicians. Historically, several papers made leading contributions toward solving the problem. H. Hopf presented some valuable machinery in a paper in 1935. G.W. Whitehead eliminated all possible values of n except $n = 2$ or $4r$ (from now on we will forget the trivial case and always assume $n > 1$) in a paper in 1950 (see references). In papers in 1952 J. Adem further eliminated the possible values of n to $n = 2^m$ using Steenrod squares in ordinary cohomology, and H. Toda eliminated the value $n = 16$. Finally J.F. Adams gave a complete solution to the problem in 1960 using a deeper analysis of Steenrod squares. We shall present here a different proof using Adams operations in K-theory.

Throughout the discussion we shall always be using a suitable category of pointed topological spaces Top_o. The multiplication problem considered in the homotopy category becomes a problem of H-spaces. The entire ensuing discussion could be considered for the homotopy category yielding similar results. That is, we would conclude that S^{n-1} is an H-space only for $n = 1, 2, 4$, and 8.

The Hopf Construction

Let us now assume we have a map (not necessarily a continuous multiplication with unit)

$$m : S^{n-1} \times S^{n-1} \to S^{n-1}.$$

We shall obtain from this the *Hopf construction*

$$\bar{m} : S^{2n-1} \to S^{n}$$

as follows. First rewrite m as

$$m : S_1 \times S_2 \to S$$

where B_1 and B_2 are n-cells such that $\partial(B_i) = S_i$. Then

$$\partial(B_1 \times B_2) = B_1 \times S_2 \cup S_1 \times B_2$$

is a (2n-1) sphere S^{2n-1}, and

$$B_1 \times S_2 \cap S_1 \times B_2 = S_1 \times S_2.$$

Let $S' = sS$ be the suspension of S (i.e. S' is the quotient space $S \times I/(S \times 0, S \times 1, e \times I)$ with base point $e \times I$). S' is an n sphere. Then $S' = H_+ \cup H_-$ where H_+ and H_- are n-cells and $H_+ \cap H_- = S$. We want to extend m to a map

$$\bar{m} : B_1 \times S_2 \cup S_1 \times B_2 \to S$$

such that $\bar{m}(B_1 \times S_2) \subset H_+$ and $\bar{m}(S_1 \times B_2) \subset H_-$. Clearly this is possible. We can make things explicit by defining the $S_1 * S_2$. This is the quotient space $S_1 \times S_2 \times I/(S_1 \times y \times 0, x \times S_2 \times 1, e \times e \times I)$ with base point $e \times e \times I$. Then

Lemma. $S^{n-1} * S^{n-1} \simeq B_1 \times S_2 \cup S_1 \times B_2$.

Proof. The map f defined by

$$f(x,y,t) = (2 \min (1-t,\tfrac{1}{2})x,\ 2 \min (t,\tfrac{1}{2})y)$$

is easily seen to be a continuous bijection between compact spaces, hence a homeomorphism. QED.
Then define the Hopf construction as the map

$$\bar{m} : S_1 * S_2 \to S'$$

given by

$$\bar{m}(x,y,t) = (m(x,y),t).$$

The Hopf Invariant

Next we shall want to define the Hopf invariant of this map $\bar{m}$. We shall need K-theory to do this, so let's first review the brief amount of K-theory needed here. We shall be using the complex K functor: Top $\to$ Rings and the complex reduced K functor $\tilde{K}$: $Top_o \to$ Rings. Given any map

$$f : X \to Y$$

we have the Puppe sequence

$$X \xrightarrow{f} Y \to C_f \to sX \to sY$$

where C_f is the *mapping cone of f* given by the quotient space $X \times I \cup Y/(X \times 0,\ x \times 1 = f(x),\ x_o \times I)$ with base point $x_o \times I$. This gives rise to the exact sequence

$$\tilde{K}(sY) \to \tilde{K}(sX) \to \tilde{K}(C_f) \to \tilde{K}(Y) \to \tilde{K}(X).$$

There is defined a cap product

$$\tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \wedge Y)$$

where the smash product $X \wedge Y$ is the quotient space $X \times Y/X \vee Y$ (the wedge product $X \vee Y$ is the quotient space $X \cup Y/(x_o = y_o)$). We then have the composition

$$\tilde{K}(X/A) \otimes \tilde{K}(X/B) \to \tilde{K}(X/A \wedge X/B) = \tilde{K}(XxX/AxX \cup XxB) \overset{\Delta^*}{\to} \tilde{K}(X/A \cup B)$$

where $\Delta : X \to XxX$ is the diagonal map. Note, finally, that $\tilde{K}(S^n) = 0$ if n is odd. $\tilde{K}(S^2)$ is the free abelian group with generator $\beta_2 = [H] - [1]$, where H is a line bundle, subject to the single relation $\beta_2^2 = 0$. For general even dimensional spheres $\tilde{K}(S^{2n})$ is the free abelian group with generator $\beta_2 \otimes \beta_2 \otimes \cdots \otimes \beta_2$ (n times) $= \beta_{2n}$.

Now we can proceed to define the Hopf invariant of $\bar{m} : S^{2n-1} \to S^n$. The Puppe sequence

$$S^{2n-1} \overset{\bar{m}}{\to} S^n \to C_{\bar{m}} \to S^{2n} \to S^{n+1}$$

gives rise to the exact sequence

Choose $x,y \in \tilde{K}(C_{\bar{m}})$ such that $\theta(\beta_{2n}) = y$ and $\phi(x) = \beta_n$. Since θ and ϕ are ring homomorphisms $\phi(x^2) = \beta_n^2 = 0$ and $x^2 = H(\bar{m})y$ for some integer $H(\bar{m})$. Define $H(\bar{m})$ as the *Hopf invariant* of $\bar{m}$. We need first to show that $H(\bar{m})$ is well-defined (i.e. independent of the x chosen). So let $\phi(x') = \beta_n$. Then $x' = x + ry$ for some integer r. Hence $(x')^2 = x^2 + 2rxy + r^2y^2 = x^2 = H(\bar{m}).y$. For, clearly $y^2 = 0$. To show $x.y = 0$, consider the two cases. If n is odd, $x = ty$ for some integer t and $x.y = ty^2 = 0$. If n is even, θ is a monomorphism. So $x.y = uy$ for some integer u and $u^2y = x^2y = H(\bar{m})y^2 = 0$, and $u = 0$. So $H(\bar{m})$ is well-defined. Note that if n is odd, $H(\bar{m}) = 0$.

The Multiplication Problem

We now quickly outline the solution to the multiplication problem for spheres. Given any map

$$m : S^{n-1} \times S^{n-1} \to S^{n-1}$$

(not necessarily a continuous multiplication with unit) we define the *bidegree* of m to be the ordered pair of integers (d_1,d_2) where d_1 is the degree of the map

$$m(\ ,y_o) : S^{n-1} \times y_o \to S^{n-1}$$

and d_2 is the degree of the map

$$m(x_o,\ ) : x_o \times S^{n-1} \to S^{n-1}.$$

From now on we shall assume $n > 1$ is even. Later we shall justify this assumption. But for now let's state the main results.

Theorem 1. Let $m : S^{n-1} \times S^{n-1} \to S^{n-1}$ be a map of bidegree (d_1,d_2). Then $H(\bar{m}) = d_1d_2$.

Theorem 2. Let $m \; S^{n-1} \times S^{n-1} \to S^{n-1}$ be a map. Then if $H(\bar{m})$ is odd, n = 2, 4, or 8.

Corollary. Let $m : S^{n-1} \times S^{n-1} \to S^{n-1}$ be a continuous multiplication with unit. Then n = 2, 4, or 8.

Proof to Corollary. Since $m(x,e) = m(e,x) = x, \forall x$, the bidegree of m is (1,1). Hence, by Theorem 1, the Hopf invariant $H(\bar{m})$ of its Hopf construction $\bar{m}$ is 1. By Theorem 2, n = 2,4, or 8. QED.

Proof to Theorem 1.

We want to prove that $H(\bar{m}) = d_1 d_2$ where (d_1,d_2) is the bidegree of $m : S^{n-1} \times S^{n-1} \to S^{n-1}$ and $\bar{m} : S^{2n-1} = B_1 \times S_2 \cup S_1 \times B_2 \to S^n = H_+ + H_-$. We have the attaching map

$$g : (B_1 \times B_2,\ B_1 \times S_2,\ S_1 \times B_2) \to (C_{\bar{m}},\ H_+,\ H_-).$$

where the mapping cone $C_{\bar{m}} = B_1 \times B_2 \cup_{\bar{m}} S^n$. Taking the cup product, using exact sequences and homotopy equivalences, we have the commutative diagram (recall, we are assuming n is even)

$$\tilde{K}(B_1 \times B_2/B_1 \times S_2) \otimes \tilde{K}(B_1 \times B_2/S_1 \times B_2) \to \tilde{K}(B_1 \times B_2/B_1 \times S_2 \cup S_1 \times B_2)$$

$$\begin{array}{ccc} \simeq \uparrow & & \simeq \uparrow \\ \tilde{K}(B_2/S_2) \otimes \tilde{K}(B_1/S_1) & & \tilde{K}(S^{2n}) \\ \simeq \uparrow & & \\ \tilde{K}(S^n) \otimes \tilde{K}(S^n). & & \end{array}$$

The image of $\beta_n \otimes \beta_n$ under this composition is β_{2n}. Let a_1 be the image of β_n in $\tilde{K}(B_1 \times B_2/S_1 \times B_2)$ and a_2 the image of β_n in $\tilde{K}(B_1 \times B_2/B_1 \times S_2)$. Then the cup product $a_1 \otimes a_2$ is the generator of $\tilde{K}(B_1 \times B_2/B_1 \times S_2 \cup S_1 \times B_2)$ which projects to β_{2n}. Using exact sequences, homotopy equivalences, excision properties, and the attaching map, we have the commutative diagram

$$
\begin{array}{ccccccc}
0 \to \tilde{K}(S^{2n}) \to & \tilde{K}(C_{\bar{m}}) & \to & \tilde{K}(S^n) & \to 0 & & \\
& \simeq \downarrow & & \simeq \downarrow & & & \\
& \tilde{K}(C_{\bar{m}}/H_-) & \to & \tilde{K}(S^n/H_-) & & & \\
& || & & \simeq \downarrow & & & \\
& \tilde{K}(C_{\bar{m}}/H_-) & \to & \tilde{K}(H_+/S^{n-1}) & \overset{\simeq}{\to} & \tilde{K}(S^n) & \\
& g^* \downarrow & & g^* \downarrow & & g^* \downarrow & \\
& \tilde{K}(B_1 \times B_2/S_1 \times B_2) & \overset{\simeq}{\to} & \tilde{K}(B_1/S_1) & \overset{\simeq}{\to} & \tilde{K}(sS_1). &
\end{array}
$$

So g^* is multiplication by d_1, and the image of x in $\tilde{K}(B_1 \times B_2/S_1 \times B_2)$ is d_1a_1. From a similar diagram the image of x in $\tilde{K}(B_1 \times B_2/B_1 \times S_2)$ is d_2a_2. Taking cup products, x^2 corresponds to $d_1d_2a_1 \otimes a_2$ in $\tilde{K}(B_1 \times B_2/B_1 \times S_2 \cup S_1 \times B_2)$ which projects to $d_1d_2\beta_{2n}$ in $\tilde{K}(S^{2n})$. But $x^2 = H(\bar{m})y$ also corresponds to $H(\bar{m})\beta_{2n}$ in $\tilde{K}(S^{2n})$. Hence $H(\bar{m}) = d_1d_2$. QED.

Proof to Theorem 2

To prove Theorem 2, we shall need to discuss Adams operations in K-theory. An *operation* in K-theory is a natural transformation $F : K \to K$ where K is regarded as a set-valued functor. We shall first define some operations in K-theory [illegible] exterior power functor λ^i defined on finitedimensional vector spaces by $\lambda^i(V) = V \wedge V \wedge \ldots \wedge V$ (i times).

They define natural transformations λ^i : Vect → Vect where Vect : Top → Ens. is the contravariant functor given by

X → set of isom. classes of complex vector bundles over X

f : X → Y → f^*, the pullback.

We wish to extend these to operations λ^i : K → K. To do this we use a neat trick. Define $\lambda_t[V] \in K(X)[[t]]$, $[V] \in \mathrm{Vect}(X)$, to be the power series

$$\sum_{i=0}^{\infty} t^i[\lambda^i[V]].$$

Since $\lambda^i(V \oplus W) = \sum_{j+k=i} \lambda^j(V) \otimes \lambda^k(W)$,

$$\lambda_t[V \oplus W] = \lambda_t[V]\lambda_t[W].$$

Also note that each $\lambda_t[V]$ is a unit in K(X)[[t]] since it has constant lending term 1. Thus we have a homomorphism

$$\lambda_t : \mathrm{Vect}\ (X) \to 1 + K(X)[[t]]^+$$

of the additive semigroup Vect (X) into the multiplication group of power series over K(X) with constant term 1. By the universal property of K(X), this extends uniquely to a homomorphism

$$\lambda_t : K(X) \to 1 + K(X)[[t]]^+$$

taking the coefficients of t^i, we have defined the operations

$$\lambda^i : K(X) \to K(X).$$

Note that if L is a line bundle

$$\lambda_t[L] = [1] + t[L].$$

So $\lambda^o[L] = [1]$, $\lambda^1[L] = [L]$, and $\lambda^i[L] = [0]$, $i > 1$.

Now we can define the Adams operations $\Psi^i : K \to K$. Let $\Psi^o(x) =$ rank x (component-wise the trivial bundle with dimension equal to that of a fiber). Define

$$\Psi_t(x) \in K(X)[[t]],\ x \in K(X), \text{ by}$$

$$\Psi_t(x) = \Psi^o(x) - t \frac{d}{dt}(\log \lambda_{-t}(x)).$$

Then define the Adams operations Ψ^i as the coefficients of the t^i; i.e.,

$$\Psi_t(x) = \sum_{i=0}^{\infty} t^i \Psi^i(x).$$

Lemma. The Adams operations $\Psi^i : K \to K$ satisfy the following properties:

(1) $\Psi^i(x + y) = \Psi^i(x) + \Psi^i(y)$ $x,y \in K(X)$.

(2) If L is a line bundle, $\Psi^i(L) = L^i$.

Properties (1) and (2) uniquely characterize the operations Ψ^i. In addition,

(3) $\Psi^i(xy) = \Psi^i(x)\ \Psi^i(y)$, $x,y \in K(X)$.

(4) $\Psi^k\Psi^j(x) = \Psi^{kj}(x)$, $x \in K(X)$.

(5) If p is prime, $\Psi^p(x) \equiv x^p \mod p$.

(6) If $u \in \tilde{K}(S^{2n})$, $\Psi^i(u) = i^n u$.

Proof. Since $\Psi_t(x + y) = \Psi_t(x) + \Psi_t(y)$, (1) follows. We have shown that $\lambda_{-t}(L) = 1 - tL$ where L is any line bundle, hence

$$\Psi_t(L) = 1 - t(\frac{-L}{1-tL}) = 1 + tL(\sum_{i=0}^{\infty} (tL)^i)$$

$$= \sum_{i=0}^{\infty} t^i L^i$$

and (2) is proved. That properties (1) and (2) uniquely characterize the operations Ψ^i follows immediately from Lemma (Splitting Principle). Let $E_i \to X$, $1 \leq i \leq n$, be complex vector bundles over X. Then $\exists$ map $\pi : Y \to X \ni$

(a) $\pi^* : K(X) \to K(Y)$ is injective

(b) each $\pi^* (E_i)$ is a direct sum of line bundles.

Similarly (3), (4), and (5) follow from the Splitting Principle. Finally, in $\tilde{K}(S^2)$, $\Psi^i(\beta_2) = i\beta_2$, so, applied to the generator $\beta_{2n} = \beta_2 \otimes \beta_2 \otimes \cdots \otimes \beta_2$ (n times) of $\tilde{K}(S^{2n})$, we get

$$\Psi^i(\beta_{2n}) = i^n \beta_{2n}$$

and hence (6). QED.

Note that the Adams operations restrict to $\tilde{K}$. Then we are in a position to prove Theorem 2. Since we are assuming n is even, we have the short exact sequence

$$0 \to \tilde{K}(S^{4n}) \overset{\Theta}{\to} \tilde{K}(C_{\underline{m}}) \overset{\phi}{\to} \tilde{K}(S^{2n}) \to 0$$

where $\Theta(\beta_{4n}) = y$, $\phi(x) = \beta_{2n}$, and $x^2 = H(\bar{m})y$. We shall apply some Adams operations to x and y. Note that

$$\Psi^2(x) = 2^n x + ay, \ a \in Z,$$

$$\Psi^3(x) = 3^n x + by, \ b \in Z$$

and

$$\Psi^k(y) = k^{2n} y$$

by property (6) of the Adams operations. But by property (5),

$$\psi^2(x) \equiv x^2 \bmod 2 \equiv H(\bar{m})y \bmod 2.$$

So, since $H(\bar{m})$ is odd, a must be odd. Hence, by property (4),

$$\psi^6(x) = \psi^3\psi^2(x) = \psi^3(2^n x + ay)$$
$$= 2^n 3^n x + 2^n by + 3^{2n} ay$$

and

$$\psi^6(x) = \psi^2\psi^3(x) = \psi^2(3^n x + by)$$
$$= 3^n 2^n x + 3^n ay + 2^{2n} by.$$

Thus

$$3^n a(3^n - 1) = 2^n b(2^n - 1),$$

and, since a is odd,

$$2^n/3^n - 1.$$

By elementary number theory this can happen only if $n = 1, 2,$ or 4. QED.

Ordinary Cohomology

The multiplication problem was originally solved using Steenrod's equivalent definition of Hopf invariant given in terms of ordinary cohomology. We shall discuss this approach. Our aim will be to obtain our previous assumption that n must be even. We shall also outline Adem's proof that n must be a power of 2.

So, let H^* be reduced singular cohomology with coefficients in G. The map

$$m : S^{n-1} \times S^{n-1} \to S^{n-1}, \; n > 1,$$

gives rise to its Hopf construction

$$\bar{m} : S^{2n-1} \to S^n.$$

Looking at the exact sequence of the pair $(C_{\bar{m}}, S^n)$, we see that

$$H^r(C_{\bar{m}}) = \begin{cases} G & r = n,\ 2n \\ 0 & \text{otherwise.} \end{cases}$$

Let G be either Z or Z_2 and let x be the generator of $H^n(C_{\bar{m}})$ and y the generator of $H^{2n}(C_{\bar{m}})$. Then taking the cup product

$$x^2 = H(\bar{m})y$$

for some integer $H(\bar{m})$ defined to be the *Hopf invariant*. For this ordinary cohomology definition of Hopf invariant, Theorem 1 holds $\forall$ $n > 1$ (odd or even).

Theorem 1. Let $m : S^{n-1} \times S^{n-1} \to S^{n-1}$ be a map of bidegree (d_1, d_2). Then $H(\bar{m}) = d_1 d_2$.

Proof. Completely analogous to the K-theory proof, only using the exact sequence

$$0 \to H^n(C_{\bar{m}}) \xrightarrow{\cong} H^n(S^n) \to 0. \quad \text{QED.}$$

Note by the commutativity properties of the cup product that

$$x^2 = (-1)^{n^2} x^2.$$

So letting $G = Z$, if n is odd, $2x^2 = 0$ and $H(\bar{m}) = 0$. We have thus reduced the multiplication problem to a consideration of n even.

We can further reduce the values of n under consideration to powers of 2; i.e. $n = 2^k$. This result by Adem uses Steenrod squares. We shall outline the proof. From now on assume $G = Z_2$.

Theorem A. There exist unique Steenrod square operations Sq^i: $H^r(X,A) \to H^{r+i}(H,A)$, $i \geq 0$, which are homomorphisms and have the following properties

(1) $Sq^0 = id$.
(2) If $\dim x = i$, $Sq^i(x) = x^2$.
(3) If $i > \dim x$, $Sq^i(x) = 0$.
(4) Cartan formula:

$$Sq^i(xy) = \sum_{j+k=i} Sq^j(x).Sq^k(y).$$

(5) Sq^1 is the Bochstein homomorphism β of the exact coefficient sequence

$$0 \to Z_2 \to Z_4 \to Z_2 \to 0.$$

(6) Adem relations: if $0 \leq a < 2b$, then

$$Sq^a.Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j}.Sq^j.$$

Define R(2), the Steenrod algebra mod 2, to be the graded associative algebra generated by the Sq^i. In detail, let M be the graded Z_2-module with $M_i = Z_2$. Denote the generator of M_i as Sq^i. R(2) is the quotient of the tensor algebra $\Gamma(M)$ by relations of the form

$$Sq^a.Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j}Sq^j, \ 0 \leq a < 2b.$$

Theorem B. The elements Sq^{2^k} generate R(2) as an algebra.

Corollary. Let $m : S^{n-1} \times S^{n-1} \to S^{n-1}$ be a continuous multiplication with unit, $n > 1$. Then $n = 2^k$.

Proof. We have the relation $x^2 = H(\bar{m}).y$ in $H^*(C_{\bar{m}})$. By Theorem 1, $H(\bar{m}) = 1$. Hence

$$Sq^n(x) = x^2 \neq 0.$$

Using Theorem B, $Sq^{2^k}(x) \neq 0$ for some $k > 0$. But looking at $H^*(C_m)$, we must have $2^k = n$. QED.

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(Received 12th November 1974)