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SCHRÖDINGER EQUATION IN GENERAL RELATIVITY*)

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R I N G K A S A N

Dalam tulisan ini diturunkan persamaan Schrödinger ke dalam TRU dari suatu partikel bermuatan dalam potensial Coulomb, yang diperoleh sebagai approksimasi dari pada persamaan Klein-Gordon dalam TRU dengan anggapan bahwa energi kinetik dan potensialnya kecil sekali dibandingkan terhadap energi diam mc^2. *Potensial gravitasi-Newton muncul di dalam formulasi ini sebagaimana diharapkan dalam bentuk non-relativistiknya.*

Ditunjukkan pula disini bahwa persamaan radialnya memiliki titik $r = o$ *sebagai titik singular yang non-essensial, yang mana memberikan eksistensi solusinya dalam penguraian deret sekitar* $r = o$. *Solusi approksimasi dengan mempergunakan teori Perturbasi ternyata memunculkan beberapa divergensi yang masih belum terpecahkan dalam tulisan ini.*

A B S T R A C T

Schrödinger equation in GR for a charged particle in Coulomb potential is presented, derived from the formulation of Klein-Gordon equation in GR with an assumption that the kinetic and potential energy are small in comparison with the rest energy mc^2. *The term Newtonian potential appeared directly as ex-*

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pected from the non-relativistic Schrödinger equation in the Newtonian gravitational field.

It is shown that its radial equation posseses a non-essential singular point at r = o, therefore it has a power series solution about r = o. Perturbation theory is used to find its approximation solution, encountered some divergences that remained unsolved.

I. Introduction

Schrödinger equation that was given here is not a covariant equation as required in GR but just an approximation of a covariant Klein-Gordon equation in GR. Approximation of a covariant Klein-Gordon equation to Schrödinger equation in SR is given in Chapter II, by taking on its interaction with EM-field.

Rewritting the Klein-Gordon equation in SR into a tensor form equation, its generalization to a covariant form in Riemannian space of the solution of Einstein-Maxwell equation in GR is obtained in Chapter III, for which the problem is specialized to its interaction with Coulomb potential. Schrödinger equation in GR is obtained in Chapter IV, by making used of the method given in Chapter II.

Finally, it is shown in Chapter V that its radial equation posseses a non-essential singular point at r = o, therefore its solution could be expand as power series about r = o. It must be noted that there appeared no Schwarzschild singularity if the ratio of charged to the mass of particle that generate the field is greater than the square root of gravitational constant. Approximation with Perturbation theory encountered some divergences in its third and fourth power of 1/r, that remained unsolved in this paper.

II. Schrödinger equation as an approximation of Klein-Gordon equation in SR

Schrödinger equation that included EM potential $\vec{A}(\vec{r}, t)$ and $\emptyset(\vec{r}, t)$,

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = [- \frac{\hbar^2}{2m} (\vec{\nabla} - \frac{ie}{c\hbar} \vec{A})^2 + e\emptyset]\psi(\vec{r}, t) \qquad \text{(II.1)}$$

is a non-relativistic wave equation since it is not covariant under Lorentz transformation of SR.

This is the main obstacle for deriving it directly in GR. However, this obstacle could be over come with the condition that Schrödinger equation (II.1) is just an approximation of the relativistic Klein-Gordon equation which for the kinetic and potential energy are small in comparison with the rest energy mc^2.

Klein-Gordon equation that included EM potential is derived from the relativistic relation between energy-momentum and EM-potential,

$$(E - e\emptyset)^2 = (c\vec{p} - e\vec{A})^2 + m^2c^4 \qquad \text{(II.2)}$$

with the substitution of energy and momentum operator,

$$E = i\hbar \frac{\partial}{\partial t}; \ \vec{p} = - i\hbar\nabla \qquad \text{(II.3)}$$

to (II.2) and operated it to the wave function $\psi(\vec{r}, t)$ then the result is,

$$(- \hbar^2 \frac{\partial^2}{\partial t^2} - 2ie\hbar \ \emptyset \frac{\partial}{\partial t} - ie\hbar \frac{\partial}{\partial t} \emptyset + e^2\emptyset^2)\psi(\vec{r}, t) =$$

$$(- \hbar^2c^2\vec{\nabla}^2 + 2ie\hbar c\vec{A}.\vec{\nabla} + ie\hbar c(\text{div } \vec{A}) + e^2A^2 + m^2c^4)\psi(\vec{r}, t)$$

$$\text{(II.4)}$$

By the assumption that the total energy $E = E' + mc^2$, were E', $\emptyset \ll mc^2$, gives $(E - e\emptyset)^2 \approx 2mc^2(E' - e\emptyset) + m^2c^4$ then substituted to (II.2) where m^2c^4 term canceled and then divide with $2mc^2$, the Schrödinger equation in EM-field is obtained,

$$E'\psi(\vec{r}, t) = [- \frac{\hbar^2}{2m} (\vec{\nabla} - \frac{ie}{c\hbar} \vec{A})^2 + e\emptyset] \ \psi \ (\vec{r}, t) \qquad \text{(II.5)}$$

the right side in the bracket is the non-relativistic Hamiltonian operator of the charge particle in the EM-field. Substitution of energy operator in the left side gives Schrödinger equation (II.1). The reader should be realized that the wave-function $\psi(\vec{r}, t)$ in (II.4) and (II.5) are not identical.

This is the method that should be applied to derive the Schrödinger equation in GR.

III. Klein-Gordon equation in GR

Klein-Gordon equation (II.4) would be simplified in tensor form and then generalized it to the GR. Mathematically, this is just to transfer physical equation in Minkowsky four dimensional spacetime which characterized by the metric $ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$ (where $\eta_{\mu\nu}$ is Lorentzian metric tensor with $\eta_{oo} = -\eta_{11} = -\eta_{22} = -\eta_{33} = 1$ and $\eta_{\mu\nu} = 0$ if $\mu \neq \nu$ and $x^\mu(\mu = 0, 1, 2, 3)$ is the spacetime coordinate where the index 0 denoted time coordinate $x^o = ct$), into Riemannian four dimensional spacetime, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ (where $g_{\mu\nu} = g_{\mu\nu}(x^\rho)$ the general metric tensor). We have used the notation that the repeated indices denoted summation over them.

Introducing the four vector potential of EM-field, Φ_μ where Φ_i $(i = 1, 2, 3) = \vec{A}$ and $\Phi_o = -\emptyset$, we obtain the tensor equation of (II.4).

$$\eta^{\mu\nu}\psi_{,\mu\nu} - \frac{2ie}{\hbar c}\,\Phi^\mu\psi_{,\mu} - \frac{ie}{\hbar c}\,\Phi^\mu{}_{,\mu}\psi - \frac{e^2}{c^2\hbar^2}\,\Phi^\mu\Phi_\mu\psi + \left(\frac{mc}{\hbar}\right)^2\psi = 0 \qquad \text{(III.1)}$$

where the comma's after a quantity denoted partial derivative. It is generalized to covariant derivative in Riemannian space which denoted with ";" mark after a quantity, which is identical with partial derivative for a scalar function. The operation $\Phi^\mu{}_{,\mu}$ is the divergence operation in Minkowsky spacetime that was generalized in Riemannian space as,

$$\Phi^\mu{}_{;\mu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\;\Phi^\mu\right)_{,\mu} \qquad g = \det(g_{\mu\nu}) < 0$$

Furthermore, the operation $\eta^{\mu\nu}\psi_{,\mu\nu}$ is $(\text{Grad})^2\psi$ operation in

Minkowsky space where its generalization to Riemannian space is given by,

$$g^{\mu\nu}\psi_{;\mu\nu} = \frac{1}{\sqrt{-g}} (\sqrt{-g}\ g^{\mu\nu}\psi_{,\mu})_{,\nu}$$

Therefore, the generalization of Klein-Gordon equation (II.4) to the Riemannian spacetime is,

$$\frac{1}{\sqrt{-g}} (\sqrt{-g}\ g^{\mu\nu}\psi_{,\mu})_{,\nu} - \frac{2ie}{\hbar c} \Phi^{\mu}\psi_{,\mu} - \frac{ie}{\hbar c} \frac{1}{\sqrt{-g}} (\sqrt{-g}\ \Phi^{\mu})_{,\mu} +$$

$$- \frac{e^2}{c^2\hbar^2} \Phi^{\mu}\Phi_{\mu}\psi + (\frac{mc}{\hbar})^2\psi = 0 \qquad \text{(III.2)}$$

The generalized Lorentz condition for EM potential Φ_{μ},

$$\Phi^{\mu}{}_{;\mu} = 0 \qquad \text{(III.3)}$$

reduced the above equation to,

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}\ g^{\mu\nu}\psi_{,\mu})_{,\nu} - \frac{2ie}{\hbar c}\Phi^{\mu}\psi_{,\mu} - \frac{e^2}{c^2\hbar^2}\Phi^{\mu}\Phi_{\mu}\psi + (\frac{mc}{\hbar})^2\psi = 0 \qquad \text{(III.4)}$$

Since we discussed the Klein-Gordon equation which interact with the EM-field, the Riemannian spacetime that should be used here is the solution of Einstein-Maxwell field equation,

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = (\frac{8\pi G}{c^4})\ E_{\mu\nu}$$

$$F_{\mu\nu;\rho} + F_{\nu\rho;\mu} + F_{\rho\mu;\nu} = 0 \qquad \text{(III.5)}$$

$$F_{\mu}{}^{\nu}{}_{;\nu} = 0$$

where G is the gravitational constant,

$$E_{\mu\nu} = (4\pi)^{-1}(F_{\mu\rho}\, F^{\rho}_{\nu} - \tfrac{1}{4}g_{\mu\nu}\, F_{\rho\sigma}\, F^{\rho\sigma})$$

is the momentum - energy density tensor of the EM-field and $F_{\mu\nu} = \Phi_{\mu;\nu} - \Phi_{\nu;\mu}$ is the EM-field tensor. $R_{\mu\nu}$ and R are respectively contracted tensors of the Riemannian Curvature tensor $R_{\mu\nu} = R_{\sigma\mu\nu,}{}^{\sigma}$ and $R = g^{\mu\nu}R_{\mu\nu}$

IV. Schrodinger equation with Coulomb potential in GR

Furthermore, the problem is specialized to the case in which $\Phi_i = 0$ and $\Phi_o = -\emptyset(r)$ which is spherical symmetry and static that is the Coulomb potential. This condition simplified the solution of Einstein-Maxwell equation (III.3), which its solution is given by the Scwarzschild metric,

$$ds^2 = -(1 - \frac{A}{r} + \frac{B}{r^2})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\emptyset^2)$$

$$+ (1 - \frac{A}{r} + \frac{B}{r^2})c^2dt^2 \qquad \text{(IV.1)}$$

$$A = \frac{2GM}{c^2}; \qquad B = \frac{Gq^2}{c^4}$$

where M and q are mass and charge of the particle that generate the field respectively. It seem that Schwarzschild singularity does not exist in this solution, if $(q/M) > G^{\frac{1}{2}}$.

Substitution of the metric tensor (IV.1) into (III.4) and replacing the time differentiation with the operator E, (III.4) reduced to the form of Klein-Gordon equation with Coulomb potential in GR,

$$(1 - \frac{A}{r} + \frac{B}{r^2})^{-1}(E - e\emptyset(r))^2\psi + \frac{\hbar^2c^2}{r^2}\frac{\partial}{\partial r}[r^2(1 - \frac{A}{r} + \frac{B}{r^2})\frac{\partial\psi}{\partial r}] \quad +$$

$$+ \frac{\hbar^2 c^2}{r^2 \sin\theta} \frac{\partial}{\partial\theta}(\sin^2\theta \frac{\partial\psi}{\partial\theta}) + \frac{\hbar^2 c^2}{r^2 \sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2} - m^2 c^4 \psi = 0 \qquad (IV.2)$$

By multiplying the both side with $(1 + \frac{A}{r} + \frac{B}{r^2})$, gives

$$(E - e\phi(r))^2\psi + (1 - \frac{A}{r} + \frac{B}{r^2}) \{\frac{\hbar^2 c^2}{r^2} \frac{\partial}{\partial r} [r^2 (1 - \frac{A}{r} + \frac{B}{r^2}) \frac{\partial\psi}{\partial r}]$$

$$+ \frac{\hbar^2 c^2}{r^2 \sin\theta} \frac{\partial}{\partial\theta} (\sin^2\theta \frac{\partial\psi}{\partial\theta}) + \frac{\hbar^2 c^2}{r^2 \sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2} - m^2 c^4 \psi\} = 0 \qquad (IV.3)$$

Bu further substitution to the total energy E with the same assumption before in Chapter II and apply the same method, we obtain the *Schrödinger equation with Coulomb interaction in GR*, given by

$$E'\psi = -\frac{\hbar^2}{2m}(1 - \frac{A}{r} + \frac{B}{r^2}) \{\frac{1}{r^2} \frac{\partial}{\partial r}[r^2 (1 - \frac{A}{r} + \frac{B}{r^2}) \frac{\partial\psi}{\partial r}] +$$

$$+ \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}(\sin\theta \frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2}\} - (\frac{GmM}{r})\psi +$$

$$+ \frac{C}{r^2} \psi + e\phi(r)\psi,$$

$$C = \frac{Gmq^2}{c^2} \quad C = \frac{Gmq^2}{c^2} \qquad (IV.4)$$

Newtonian gravitational potential (-GmM/r) that appeared in (IV.4) is just the consequence of the approximation that we used above. If we neglect all the terms with A, B,and C since all of them are small values, we get the non - relativistic Schrödinger equation with Coulomb potential in Newtonian gravitational potential.

V. Solution of Schrödinger equation in GR

Solution of (IV.4) is constructed by the method of the separation of the variables $\psi(r, \theta, \phi) = R(r)\, H(\theta)\Phi(\phi)$ which gives that the angular equations H (θ) and $\Phi(\emptyset)$ take the same form in QM, but its radial equation are modified by the GR to the form,

$$(1 - \frac{A}{r} + \frac{B}{r^2})\frac{1}{r^2}\frac{d}{dr}[r^2(1 - \frac{A}{r} + \frac{B}{r^2})\frac{dR}{dr}] + [\frac{2m}{\hbar^2}(E' - e\emptyset(r) + \frac{GmM}{r} + - \frac{C}{r^2}) - (1 - \frac{A}{r} + \frac{B}{r^2})\frac{\lambda}{r^2}]R = 0, \qquad \lambda = \ell(\ell+1), \quad \ell = 0, 1, 2, ..$$

(V.1)

By further simplification of differentiation on $(1 - \frac{A}{r} - \frac{B}{r^2})$ term, eq. (V.1) reduced to the radial Schrödinger equation and the rest term is the perturbation terms, that is

$$E'R = - \frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dR}{dr}) + e\emptyset(r)R + \frac{\lambda\hbar^2}{2mr^2}R - \frac{\hbar^2}{2m}[(\frac{A}{r^2} + \frac{2B-A^2}{r^3} + - \frac{AB}{r^4})\frac{dR}{dr} + (\frac{2B-A^2}{r^2} - \frac{2AB}{r^3} + \frac{B^2}{r^4})\frac{d^2R}{dr^2} + \frac{A\lambda}{r^3}R - \frac{B\lambda}{r^4}R] - (\frac{GmM}{r})R + + \frac{C}{r^2}R,$$

(V.2)

The question arises whether eq. (V.2) has a regular solution about the origin.

Let us multiply the both side with r^4 and divided with the cofactor of $\frac{d^2R}{dr^2}$ then rearrangement to the Fuch's type form, we get,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{2r^4 + Ar^3 + (2B-A^2)r^2 - ABr}{r^4 + (2B-A^2)r^2 - 2ABr + B^2}\frac{dR}{dr} + \frac{1}{r^2}\frac{Er^6 + GmMr^5 - Gr^2 - er^6\emptyset(r) - \frac{\lambda\hbar^2}{2m}r^4 + A\lambda r^3 - B\lambda r^2}{r^4 + (2B-A^2)r^2 - 2ABr + B^2}R = 0$$

(V.3)

or

$$\frac{d^2R}{dr^2} + \frac{1}{r} P(r)\frac{dR}{dr} + \frac{1}{r^2} Q(r)R = 0 \qquad \text{(V.4)}$$

where P(r) and Q(r) are respectively given in the above equation.

It is clear that P(r) and Q(r) are both analytic function at r = o, therefore according to Fuch's Theorem eq. (V.2), the radial equation of Schrödinger equation in GR has a regular solution at r = o, hence could be developed as a power series about r = o.

The exact solution with power series solution about the origin is more complicated, however by remembering that A, B and C are small values, we take the approximation solution with the method of the Perturbation theory.

In accordance to the Perturbation theory, the non-perturbation Hamiltonian is,

$$H^o = -\frac{\hbar^2}{2m}\nabla^2 + e\phi(r) \qquad \text{(V.5)}$$

and the Hamiltonian Perturbation is given by (IV.6), that is

$$H' = -\frac{\hbar^2}{2m}\left[\left(\frac{A}{r^2} + \frac{2B-A^2}{r^3} - \frac{AB}{r^4}\right)\frac{\partial}{\partial r} + \left(\frac{2B-A^2}{r^2} - \frac{2AB}{r^3} + \frac{B^2}{r^4}\right)\frac{\partial^2}{\partial r^2}\right.$$

$$\left. + \frac{A\lambda}{r^3} - \frac{B\lambda}{r^4}\right] - \frac{GmM}{r} + \frac{C}{r^2} \qquad \text{(V.6)}$$

Which gives the GR correction to the non-perturbed energy level of the system. Unfortunately, that the term $1/r^3$ and $1/r^4$ in (V.6) encountered some divergences in its Hamiltonian perturbation matrix, that remain unsolved in this case. Although one could obtain its exact regular equation as been shown above.

Commentary to reference 2

It had been shown by Callaway J. in Phys. Rev. 112, 290 (1958), that the radial equation of Klein-Gordon and Dirac equation in GR had not posses a regular solution about r = 0. Alternatively, if one apply the algebraic form of metric tensor i.e. non-exponential form, and do the same algebraic arrangement as mentioned in Chapter V, it seems to Fuch's theorem that those equations posseses a regular solution about r = o.

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Notes on abbreviations

GR: General Relativity, SR: Special Relativity, EM: Electromagnetic field, QM: Quantum Mechanics, TRU: Theory Relativitas Umum.

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