INVESTIGATION OF THE LIMITING BEHAVIOR OF THE SEQUENCE OF DENSITY RATIOS IN A FAMILY OF NONCENTRAL t-DISTRIBUTIONS

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ICHTISAR

Misalnja X_1, X_2, ... adalah barisan variabel random jang mempunjai distribusi t jang nonsentral dengan parameter θ. Maksud dari karangan ini adalah menjelidiki sifat² limit rasio fungsi kepadatan dari X_n tersebut. Hal ini merupakan tjontoh dari matjamnja famili distribusi jang dibitjarakan dalam karangan [4]. *Djuga karangan ini merupakan pembitjaraan lebih landjut dari* [2], *dimana David & Kruskal membuktikan bahwa sequential probability ratio test dari distribusi t akan berhenti dengan kemungkinan satu. Sifat limit dari tjontoh famili distribusi disini lebih kuat dari di* [4], *dan dapat diterangkan sbb.: Misalkan $\theta_1 < \theta_2$ adalah dua parameter dan R_n (θ_1, θ_2) adalah rasio kepadatan X_n dari θ_2 dan θ_1. Maka terdapat θ_0 sehingga limit dari R_n sama dengan 0 terhadap $\theta < \theta_0$ dan sama dengan ∞ terhadap $\theta > \theta_0$. Sedang terhadap θ_0 limit infimumnja sama dengan 0 dan limit supremumnja adalah ∞. Letak θ_0 ini diantara 0 dan $\frac{1}{2}(\theta_1 + \theta_2)$, sehingga hal jang terachir ini akan menghilangkan sangkaan orang mengenai letak simetrinja θ_0 diantara θ_1 dan θ_2.*

ABSTRACT

Let X_1, X_2, ... be a sequence of noncentral t-distributed random variables with parameter θ. In this paper we investigate the limiting behavior of the density ratios of X_n. This is an example of the family of distributions discussed in [4]. *Here, we derive the results of David & Kruskall* [2] *in a slightly more general form and a stronger result is obtained, namely*:

If $\theta_1 < \theta_2$ are two parameters and R_n (θ_1, θ_2) is the density ratio of X_n, then there is θ_0 such that the limit of R_n is 0 with respect to $\theta < \theta_0$ and is ∞ with respect to $\theta > \theta_0$. With respect to $\theta = \theta_0$, the limit infimum is 0 and the limit supremum is ∞. It is also shown that θ_0 is between 0 and $\frac{1}{2}(\theta_1 + \theta_2)$.

In this paper we shall examine the limiting behavior of the sequence of density ratios if X_1, X_2, ... is a sequence of noncentral t ratios. The results obtained will be stronger than those obtained in [4] Chap. 3. Most, but not all, of these results are also implicit in [2].

Let Z_1, Z_2, ... be independent and identically distributed random variables, the common distribution being normal with mean θ and variance 1. Define for $n \geq 2$:

$$U_n = \frac{1}{n} \sum_{i=1}^{n} Z_i, \quad V_n = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (Z_i - U_n)^2} \quad \text{and} \quad X_n = U_n / V_n$$

There is no loss of generality in assuming that the Z's have variance 1. If the variance were σ^2, we substitute Z_i/σ for Z_i in the definitions of U_n and V_n, which does not alter X_n.

In [3] the family of distributions of X_n was shown to be a monotone likelihood ratio family. The sufficiency of X_n on $\mathbf{A}_n$ follows from a factorization theorem of D.R. Cox [1], if one takes in this theorem $\theta_1 = \theta$, $\theta_2 = 1$, $T_1 = X_n$, $T_2 = \sum_{i=1}^{n} (Z_i - U_n)^2$, $U_k = X_{k-1}$, $k = 1, .., n-2$, and S is the group of transformations $Z_i \to aZ_i$, where a is any positive number.

The distribution of $\sqrt{n-1}\ V_n$ is chi with $n-1$ degrees of freedom and U_n is normal $\left(\theta, \frac{1}{n}\right)$. We recall that in [4], Chap. 2 and 3 the density ratio $p_{\theta_2}^{X_n} / p_{\theta_1}^{X_n}$ was denoted by $r_n(X_n; \theta_1, \theta_2)$. In this paper we shall suppress the dependence on θ_1, θ_2, and write $r_n(X_n)$, $r_n(x)$ instead of $r_n(X_n; \theta_1, \theta_2)$, $r_n(x; \theta_2, \theta_2)$. As usual, $\theta_1 < \theta_2$. We shall show (this result is also implicit in [2] that there is a unique number θ_o such that $r_n(x) \to 0$ or ∞ according as $x < \theta_o$ or $> \theta_o$. This allows the following conclusions concerning $r_n(X_n)$:

(1) $\lim_{n\to\infty} r_n(X_n) = 0$ a.e. P_θ if $\theta < \theta_o$

(2) $\lim_{n\to\infty} r_n(X_n) = \infty$ a.e. P_θ if $\theta > \theta_o$

To show (1) and (2) we first remark that $U_n \to \theta$ a.e. P_n and $V_n \to 1$ in probability, so that $X_n \to \theta$ a.e. P_n. Suppose $\theta < \theta_o$ and ω is not in an exceptional null set so that $X_n(\omega) \to \theta$. Choose any x such that $\theta < x < \theta_o$, then there is an integer N_ω such that $X_n(\omega) \leqslant x$ if $n \geqslant N_\omega$. Since $r_n(x)$ is an increasing function of x we have for all $n \geqslant N_\omega$: $r_n(X_n(\omega)) \leqslant r_n(x)$, so that:

$$\lim_{n\to\infty} r_n(X(\omega)) \leqslant \lim_{n\to\infty} r_n(x) = 0$$

This gives (1), and (2) is obtained analogously.

The density of X_n can be found as follows:

$$P_\theta\{X_n \leqslant x\} = \int_0^\infty p^{V_n}(v)\, dv \int_{-\infty}^{vx} p_\theta^{U_n}(u)\ du$$

so that

$$p_\theta^{X_n}(x) = \int_0^\infty p^{V_n}(v)\, p^{U_n}(vx)\ v\ dv$$

After substituting the densities of V_n and U_n we have

(3) $$p_\theta^{X_n}(x) = K_n \int_0^\infty v^{n-1} \exp\left[-\frac{n}{2}\ (vx - \theta)^2 - \frac{n-1}{2}\ v^2\right] dv$$

where

$$K_n = \frac{2\left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}}{\pi\left(\frac{n-1}{2}\right)} \sqrt{\frac{n}{2\pi}}$$

or

$$p_\theta^{X_n}(x) = K_n \exp\left[-\frac{n}{2}\theta^2\right] \int_o^\infty v^{n-1} \exp\left[-\frac{n}{2}\left(1+x^2-\frac{1}{n}\right)v^2 + n\theta x v\right] dv$$

and after making the transformation $\sqrt{n\left(1+x^2-\frac{1}{n}\right)}\, v \to v$ we have

$$p_\theta^{X_n}(x) = K_n \exp\left[-\frac{n}{2}\theta^2\right] \left\{n\left(1+x^2-\frac{1}{n}\right)\right\}^{-\frac{n}{2}} \int_o^\infty v^{n-1} \exp \left[-\frac{v^2}{2} + \frac{\theta x}{\sqrt{\left(1+x^2-\frac{1}{n}\right)}}\sqrt{n}\, v\right] dv$$

Substituting θ_1 and θ_2 for θ, respectively, and taking the ratio, we obtain

$$r_n(x) = \exp\left[-\frac{n}{2}(\theta_2^2-\theta_1^2)\right] \frac{\int_o^\infty v^{n-1}\exp\left[-\frac{v^2}{2}+\frac{\theta_2 x}{\sqrt{\left(1+x^2-\frac{1}{n}\right)}}\sqrt{n}\, v\right]dv}{\int_o^\infty v^{n-1}\exp\left[-\frac{v^2}{2}+\frac{\theta_1 x}{\sqrt{\left(1+x^2-\frac{1}{n}\right)}}\sqrt{n}\, v\right]dv} \tag{4}$$

Consider the following integrals:

$$I_n(w_n) = \int_o^\infty v^{n-1} \exp\left[-\frac{v^2}{2} + \sqrt{n}\, w_n v\right] dv \tag{5}$$

and

$$J_n(w_n) = \int_{-v_n}^\infty \left(1+\frac{y}{v_n}\right)^{n-1} \exp\left[-\frac{y^2}{2} - \frac{n-1}{v_n}y\right] dy \tag{6}$$

in which

$$v_n = \tfrac{1}{2}\sqrt{n}\left\{w_n + \sqrt{w_n^2 - 4 - \frac{4}{n}}\right\} \tag{7}$$

In [2], section 2, replacing in [2] w by w_n it is shown that:

$$I_n(w_n) = \left(\frac{\sqrt{n}}{e}\right)^{n-1} \left\{\frac{v_n}{\sqrt{n}} \exp\left[\frac{1}{2}\left(\frac{v_n}{\sqrt{n}}\right)^2\right]\right\} n \frac{\sqrt{n}}{v_n} J_n(w_n) \tag{8}$$

If

$$w_n \to w \tag{9}$$

then

$$\lim_{n\to} J_n(w_n) = \left[\frac{8\pi}{4 + (\sqrt{w^2+4} - w)^2}\right]^{\frac{1}{2}} = J(w), \text{ say} \tag{10}$$

The proof is essentially contained in (2), section 2. More precisely, the paper cited proves $J_n(w) \to J(w)$, but the proof needs only a very slight modification to obtain (10). Note that if we have (9), then

$$\lim_{n\to\infty} \frac{v_n}{\sqrt{n}} = \frac{1}{2}\left(w + \sqrt{w^2+4}\right) = \alpha(w), \text{ say.} \tag{11}$$

We shall give now a useful extension of the lemma in [2], section 2:

Lemma 1. *If*

$$w_n = w + \frac{a_n}{n} \tag{12}$$

where

$$a_n \to a \tag{13}$$

then

$$I_n(w_n) \sim \left(\frac{\sqrt{n}}{e}\right)^{n-1} e^{-1} \frac{J(w)}{\alpha(w)} \exp\left[a\alpha(w)\right] \left\{\alpha(w) \exp\left[\frac{1}{2}(\alpha(w))^2\right]\right\}^{n} \tag{14}$$

Proof: Substituting (12) into (7), we compute

$$\frac{v_n}{\sqrt{n}} = \alpha(w)\left\{1 + \frac{1}{n\sqrt{w^2+4}}\left(a - \frac{1}{\alpha(w)}\right)\right\} + o\left(\frac{1}{n}\right) \tag{15}$$

and substituting of (10), (11) and (15) into (8) yields (14).

The integral in the numerator on the right hand side of (4) is of the form (5) with $w_n = \dfrac{\theta_2 x}{\sqrt{1 + x^2 - \frac{1}{n}}}$, so that w_n is clearly of the form (12). Similarly the integral in the denominator. Thus we can apply Lemma 1 to study the limiting behavior of $r_n(x)$. If we put

$$z_i = \frac{\theta_i x}{\sqrt{1 + x^2}} \qquad i = 1, 2. \tag{16}$$

then for the limiting behavior of $r_n(x)$ we can replace the integral in the numerator (denominator) in (14) with w replaced by z_2 (z_1).
We get

$$\ln r_n(x) \sim \frac{n}{2} \left\{ -\theta_1^2 + \theta_2^2 + 2 \ln \alpha(z_2) + (\alpha(z_2))^2 - 2 \ln\alpha(z_1) - (\alpha(z_1))^2 \right\}$$

multiplied by a factor that does not involve n. Thus,

$$(17) \quad \lim_{n\to\infty} \frac{1}{n} \ln r_n(x) = -\tfrac{1}{2}(\theta_2^2 - \theta_1^2) + \ln\alpha(z_2) - \ln\alpha(z_1) + \tfrac{1}{2}(\alpha(z_2))^2 - \tfrac{1}{2}(\alpha(z_1))^2$$

If we put

$$(18) \qquad \xi = \frac{x}{\sqrt{1+x^2}}$$

then from (16) we have $z_i = \xi\,\theta_i$. Furthermore, we define

$$(19) \qquad h(\theta,x) = -\tfrac{1}{2}\theta^2 + \ln\alpha(\xi\theta) + \tfrac{1}{2}(\alpha(\xi\theta))^2$$

Then we can write (17) as

$$(20) \qquad \lim_{n\to\infty} \frac{1}{n} \ln r_n(x) = h(\theta_2,x) - h(\theta_1,x)$$

The right hand side of (20) depends on x through ξ. Since by (18) ξ and x have the same sign, we see immediately from (19) that

$$(21) \qquad h(-\theta, -x) = h(\theta, x)$$

Since r (x) is a strictly increasing function of x, the same is true for the right hand side of (20) (this can also be checked directly, and this has been done in [2]). Continuity in x is obvious. We shall show that $h(\theta_2,x) - h(\theta_1,x)$ is positive if $x = \theta_2$ and negative if $x = \theta_1$. Thus, there is a unique value of x, say θ_o, with $\theta_1 > \theta_o < \theta_2$, such that $h(\theta_2,x) - h(\theta_1,x)$ is positive, 0 or negative according as $x > \theta_o$, $= \theta_o$ or $< \theta_o$. By (20) this implies that $\ln r_n(x) \to \infty$ or $-\infty$ according as $x > \theta_o$ or $< \theta_o$.

We shall show first that $h(\theta,x)$ is concave as a fucntion of θ, for fixed x, and that it has a maximum. First we differentiate the function $\ln \alpha(z) + \frac{1}{2}(\alpha(z))^2$ with respect to z, using the definition (11) of $\alpha(z)$.
We get

$$\left\{ \frac{1}{\alpha(z)} + \alpha(z) \right\} \alpha'(z) = \sqrt{z^2+4}\;\alpha'(z) = \alpha(z)$$

We use this in differentiating $h(\theta,x)$ twice with respect to θ:

$$(22) \qquad \frac{\delta h}{\delta\theta} = -\theta + \xi\alpha(\xi\theta)$$

$$(23) \qquad \frac{\delta^2 h}{\delta\theta^2} = -1 + \xi^2\,\alpha'(\xi\theta)$$

Now $\xi^2 < 1$ and $|\alpha'| < 1$, from which it follows that $\frac{\delta^2 h}{\delta \theta^2} < 0$ for all θ. Therefore, h is strictly concave in θ. Setting the right hand side of (22) equal to 0 gives the solution $\theta = \frac{\xi}{\sqrt{1 - \xi^2}} = x$, so that this is the unique maximum Hence

(24) $$h(\theta,\ \theta) < h(\theta',\ \theta) \quad \text{if } \theta \neq \theta'$$

In particular, if we take $x = \theta_2$ on the right hand side of (20) we get $h(\theta_2,\theta_2) - h(\theta_1,\theta_2)$ which is > 0 by (24) and if we take $x = \theta_1$ we get $h(\theta_2,\theta_1) - h(\theta_1,\theta_1)$ which is < 0, as was to be shown. As was remarked before, this implies the existence of a unique θ_o such that

(25) $$h(\theta_2,\theta_o) - h(\theta_1,\theta_o) = 0$$

We shall sometimes denote the solution of (25) by $\theta_o(\theta_1,\theta_2)$. From (21) we see that if the triple $(\theta_1,\theta_2,\theta_o)$ satisfies (25), then so does the triple $(-\theta_2, -\theta_1, -\theta_o)$. We can express this by

(26) $$\theta_o(-\theta_2,-\theta_1) = -\theta_o(\theta_1,\theta_2)$$

Since $\alpha(0) = 1$ we have from (19): $h(\theta_2,0) - h(\theta_1,0) = -\frac{1}{2}(\theta_2^2 - \theta_1^2) = -\frac{1}{2}(\theta_2 - \theta_1)(\theta_2 + \theta_1)$, so that $\theta_1 + \theta_2$ and $h(\theta_2,0) - h(\theta_1,0)$ have opposite sign. This, together with the fact that $h(\theta_2,x) - h(\theta_1,x)$ is increasing in x implies

(27) $$\theta_o > 0 \qquad \text{if } \theta_1 + \theta_2 > 0$$

(28) $$\theta_o = 0 \qquad \text{if } \theta_1 + \theta_2 = 0$$

(29) $$\theta_o < 0 \qquad \text{if } \theta_1 + \theta_2 < 0$$

as shown also in (2). Ofcourse, (28) and (29) are also a consequence of (27) and (26).

Now we are going to show more about the position of θ_o than is given by (27), (28) and (29), namely

(30) $$0 < \theta_o < \tfrac{1}{2}(\theta_1 + \theta_2) \qquad \text{if } \theta_1 + \theta_2 > 0$$

(31) $$0 > \theta_o > \tfrac{1}{2}(\theta_1 + \theta_2) \qquad \text{if } \theta_1 + \theta_2 < 0$$

This can be expressed in words by saying that θ_o lies between 0 and $(\theta_1 + \theta_2)/2$ which also covers the case $\theta_o = 0$ if $\theta_1 + \theta_2 = 0$. We need only show (30) since (31) follows from (30) and (26). Moreover, the left inequality in (30) is the same as (27), so that we only have to show $\theta_o < (\theta_1 + \theta_2)/2$. Define

(32) $$g(\eta) = h(x + \eta;\ x) - h(x - \eta;\ x)$$

Actually, g also depends on x, but this is suppressed in the notation. We need to consider g only for $x < 0$ and $\eta \geq 0$. Using (23) we compute

(33) $$g''(\eta) = \xi^2\, \alpha'(\xi(x + \eta)) - \alpha'(\xi(x - \eta))$$

By (18) ξ and x have the same sign, so $\xi < 0$. Since $\alpha'(w)$ is an increasing function of w the right hand side of (33) is > 0. Moreover, if follows immediately from (32) that $g(0) = g'(0) = 0$. Thus, $g(\eta) > 0$ for all $\eta > 0$, or

$$h(x + \alpha ;x) > h(x - \eta ;x) \qquad \text{if } x > 0,\ \eta > 0 \tag{34}$$

Now let θ_o satisfy (25), and substitute in (34) $x = \theta_o$, $\eta = \theta_o - \theta_1$; so that $x > 0$, $\eta > 0$. On the left in (34) we have $h(2\theta_o - \theta_1;\theta_o)$. On the right we have $h(\theta_1;\theta_o)$, which equals $h(\theta_2;\theta_o)$ by (25). Thus, (34) reduces to

$$h(2\theta_o - \theta_1;\theta_o) > h(\theta_2;\theta_0) \tag{35}$$

Now $h(\theta;\theta_o)$ as a function of θ has its maximum at $\theta = \theta_o$. Since $\theta_2 < \theta_o$, $h(\theta;\theta_o)$ is a decreasing function of θ for $\theta \geq \theta_o$. It follows then from (35) that $2\theta_o - \theta_1 < \theta_2$; or $\theta_o < (\theta_1 + \theta_2)/2$, as was to be shown.

Finally we are going to investigate the limiting behavior of $r_n(X_n)$ if $\theta = \theta_0$. Consider the following sequence

$$x_n = \theta_o + c/n \tag{36}$$

where c is some constant. If in (4), x is replaced by x_n, then the integral in the denominator of (4) is of the form (5) with $w_n = \dfrac{\theta_1 x_n}{\sqrt{1 + x_n^2 - 1/n}}$, where x_n is given by (36). This w_n is of the form (12); denote the limit on the right hand side of (13) by $a^{(1)}$. For the integral in the numerator of (4) we have a similar expression for w_n, with θ_1 replaced by θ_2, and limit in (13) given by $a^{(2)}$ We compute w_n:

$$\begin{aligned}
w_n &= \theta_i(\theta_o + c/n)\left\{1 + (\theta_o + c/n)^2 - 1/n\right\}^{-\frac{1}{2}} \\
&= \theta_i(\theta_o + c/n)(1 + \theta_o^2)^{-\frac{1}{2}}\left(1 - \frac{1}{2n}\,\frac{2c\theta_o - 1}{1 + \theta_o^2} + o(1/n)\right) \\
&= \frac{\theta_i\theta_o}{\sqrt{1 + \theta_o^2}} + \frac{1}{n}\left(\frac{\theta_i c}{\sqrt{1 + \theta_o^2}} - \frac{1}{2}\,\frac{\theta_i\theta_o(2c\theta_o - 1)}{(1 + \theta_o^2)^{3/2}} + o(1)\right) \\
&= \frac{\theta_i\theta_o}{\sqrt{1 + \theta_o^2}} + \frac{1}{n}\left(\frac{\theta_i\theta_o}{2(1 + \theta_o^2)^{3/2}} + \frac{\theta_i c}{(1 + \theta_o^2)^{3/2}} + o(1)\right)
\end{aligned}$$

so that

$$a^{(i)} = \frac{\theta_i\theta_o}{2(1 + \theta_o^2)^{3/2}} + \frac{\theta_i c}{(1 + \theta_o^2)^{3/2}} \tag{37}$$

Since $x_n \to \theta_o$, the values of z_i to be substituted for w in the asymptotic formula (14) are given by (16) $x = \theta_o$. Using (14) in (4) the factor that depends exponentially on n is then

$$\exp\left[n\left(h(\theta_2,\theta_o) - h(\theta_1,\theta_o)\right)\right]$$

which is equal to 1 for all n by (25). From the remaining factors in (14) we obtain

$$r_n(X_n) \sim \exp \left[a^{(2)} \alpha \left(\frac{\theta_2 \theta_o}{\sqrt{1+\theta_o^2}} \right) - a^{(1)} \alpha \left(\frac{\theta_1 \theta_o}{\sqrt{1+\theta_o^2}} \right) \right.$$

multiplied by a positive factor that does not involve c. Using (37) we have then

(38) $$\lim_{n \to \infty} r_n(x_n) = A \exp [Bc]$$

where A is a positive constant and

(39) $$B = \frac{\theta_2}{(1+\theta_o^2)^{3/2}} \alpha \left(\frac{\theta_2 \theta_o}{\sqrt{1+\theta_o^2}} \right) - \frac{\theta_1}{(1+\theta_o^2)^{3/2}} \alpha \left(\frac{\theta_1 \theta_o}{\sqrt{1+\theta_o^2}} \right)$$

We claim that $B > 0$. If $\theta_1 \leqslant 0 \leqslant \theta_2$ this follows immediately from the positiveness of the function α. If $0 < \theta_1 < \theta_2$, so that also $\theta_o > 0$, we make use of the fact that α is an increasing function of its argument. The case $\theta_1 < \theta_2 \leqslant 0$ is reduced to the case $0 \leqslant \theta_1 < \theta_2$ by making the transformation $\theta_1 \to -\theta_2$, $\theta_2 \to -\theta_1$, $\theta_o \to -\theta_o$ (using (26)), which leaves B invariant.

We shall use (38) now to study the limiting behavior of $r_n(X_n)$ if $\theta = \theta_o$. We know that with respect to P_{θ_o}, $\sqrt{n}\,(X_n - \theta_o)$ has a limiting distribution which is normal with mean 0 and variance 1, so that if c is a positive number

(40) $P_{\theta_o}(\theta_o - c/n \leqslant X_n \leqslant \theta_o + c/n) = P_{\theta_o}(-c/\sqrt{n} \leqslant \sqrt{n}\,(X_n - \theta_o) \leqslant c/\sqrt{n}) \to 0$ as $n \to \infty$. Let b and d be any numbers with $0 < b < d < \infty$, there is a positive number c such that

(41) $$A \exp[-Bc] < b < d < A \exp [Bc]$$

From (38) there is then an integer N such that for $n \geqslant N$ we have

(42) $$r_n(\theta_o - c/n) < b \quad \text{and} \quad r_n(\theta_o + c/n) > d$$

So we have

$$\begin{aligned} P_{\theta_o}(b \leq r_n(X_n) \leq d) &\leq P_{\theta_o}(r_n(\theta_o - c/n) < r_n(X_n) < r_n(\theta_o + c/n)) \\ &\leq P_{\theta_o}(\theta_o - c/n \leq X_n \leq \theta_o + c)n) \\ &\to 0 \text{ as } n \to \infty \qquad \text{by (40)} \end{aligned}$$

and using (40) we get

(43) $$P_{\theta_o}\{b \leq r_n(X_n) \leq d\} \to 0 \text{ as } n \to \infty$$

Finally we are going to show that

(44) $$\liminf_{n \to \infty} r_n(X_n) = 0 \quad \text{a.e. } P_{\theta_o}$$

(45) $$\limsup_{n \to \infty} r_n(X_n) = \infty \text{ a.e. } P_{\theta_o}$$

It is sufficient to show that for every positive and finite number b:

(46) $$\lim_{n \to \infty} P_{\theta_o} \{r_n(X_n) \geqslant b\} = \tfrac{1}{2}.$$

for this implies (45), and (44) is implied by $\lim_{n \to \infty} P_{\theta_o}\{r_n(X_n) \leqslant b = \}$ which follows immediately from (46). To show (46), let c be a positive number satisfying

(47) $$A \exp[-Bc] < b < A \exp[Bc]$$

Similar to the proof that led to (43), we have

(48) $P_{\theta_o}(\theta_o + c/n \leqslant X_n) \leq (P_{\theta_o}(b \leqslant r_n(X_n)) \leqslant P_{\theta_o}(\theta_o - c/n \leqslant X_n)$ for $n \geqslant N$. Since $\sqrt{n}\,(X_n - \theta_n)$ has a limiting distribution which is normal with mean 0 and variance 1, we have

(49) $P_{\theta_o}(\theta_n + c/n \leqslant X_n) = P_{\theta_o}(+ c/\sqrt{n} \leqslant \sqrt{n}\,(X_n - \theta_o)) \rightarrow \frac{1}{2}$ as $n \rightarrow \infty$

It is easy to see that (48) and (49) imply (46).

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