New Hermitian Self-Dual MDS or Near-MDS Codes over Finite Fields

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Abstract. A linear code over a finite field is called Hermitian self-dual if the code is self-dual under the Hermitian inner-product. The Hermitian self-dual code is called MDS or near-MDS if the code attains or almost attains the Singleton bound. In this paper we construct new Hermitian self-dual MDS or near-MDS codes over $GF(9)$, $GF(25)$, and $GF(121)$ of length up to 14.

Keywords: decoding error probability performance; Hermitian self-dual codes; lexicographical ordering; MDS codes; near-MDS codes.

1 Introduction

A linear $[n,k]$ code $C$ over $GF(q)$ is a $k$-dimensional subspace of $GF(q)^n$, where $GF(q)$ is the Galois field with $q$ elements. The value $n$ is called length of $C$ and every element of $C$ is called codeword of $C$. The weight $w(c)$ of a codeword $c \in C$ is the number of nonzero components of $c$. The minimum weight $d$ of all nonzero codewords in $C$ is called minimum weight of $C$. An $[n,k,d]$ code is an $[n,k]$ code with minimum weight $d$. The weight enumerator $W$ of $C$ is given by

$$W(y) = \sum_{k=0}^{n} A_k y^k,$$

where $A_k$ denotes the number of codewords of weight $k$ in $C$.

The space $GF(q)^n$ is equipped by Hermitian inner-product defined by

$$[x, y] = \sum_{k=1}^{n} \overline{x_k} y_k,$$
for two vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $GF(q)^n$, where $y_k = y_k^q$, and $q = p^m$, for a prime number $p$ and an even $m$.

The Hermitian dual code $C^\perp$ of $C$ is defined as

$$C^\perp = \{ x \in GF(q)^n : [x, c] = 0, \forall c \in C \}.$$ 

A code $C$ is called Hermitian self-dual if $C = C^\perp$. From now on, what we mean by self-dual is Hermitian self-dual.

A linear $[n, k, d]$ code over $GF(q)$ satisfies the Singleton bound $d \leq n - k + 1$ (see, e.g., [1]). If the equality is attained then the code is called MDS code. The $[n, k, n-k]$ code is called almost MDS code [2]. An $[n, k, n-k]$ almost MDS code for which the dual code is also an almost MDS is called near-MDS code [3].

MDS codes are important in Mathematics since they are equivalent to geometric objects called $n$-arcs [1, p. 326] and also to combinatorial objects called orthogonal arrays [1, p. 326]. Moreover, very recently, Dodunekov [4] and Zhou, et al. [5] announced the importance of self-dual near-MDS codes in Cryptography, in particular in secret sharing schemes. Hence there is a great interest in the construction of MDS or near-MDS self-dual codes over finite fields (see, e.g., [6-10]).

Kim and his co-authors ([8,10]) used a construction method, called the building-up method, to construct self-dual MDS or near-MDS codes. They also showed that every self-dual codes over certain fields can be obtained by their building-up method. In particular, [8] provided three examples, one example, of self-dual near-MDS codes of length 12 over $GF(9)$, $GF(25)$, respectively. Recently, Gulliver, et al. [10] gave an example of self-dual MDS code of length 14 and stated that they also found many self-dual near-MDS codes of length 16 over $GF(121)$. From the generator matrix of self-dual near-MDS of length 14 above, they [10] found one self-dual MDS code of length 12, 10, 8, 6, 4, and 2, respectively.

The purpose of this paper is to provide some more examples of MDS or near-MDS self-dual codes. We obtained several new MDS or near-MDS self-dual codes of length 10 and 12 over $GF(9)$, 10, 12, and 14 over $GF(25)$, and 4, 6, 8, and 10 over the field $GF(121)$ which were unknown to exist before.
2 Construction Method

We use the following building-up construction given in [8].

**Theorem 2.1** Let $G_0 = (L | R) = (l_i | r_i)$ be a generator matrix of a self-dual code $C_0$ over $GF(q^2)$ of length $2n$, where $l_i$ and $r_i$ are the rows of the matrices $L$ and $R$ respectively, for $1 \leq i \leq n$. Let $x = (x_1, ..., x_n, x_{n+1}, ..., x_{2n})$ be a vector in $GF(q^2)^{2n}$ with $[x, x] = -1$ in $GF(q^2)$. Set $y_i = [(x_1, ..., x_n, x_{n+1}, ..., x_{2n}), (l_i | r_i)]$ for $1 \leq i \leq n$, and $c = \zeta^{q-1}$ for $(q^2 - 1)$-th root of unity $\zeta$ in $GF(q^2)$ (and hence $cc = -1$). Then the matrix

$$
\begin{pmatrix}
1 & 0 & x_1 & \cdots & x_n & x_{n+1} & \cdots & x_{2n} \\
-y_1 & cy_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-y_n & cy_n & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix} L \quad R
$$

generates a self-dual code $C$ over $GF(q^2)$ of length $2n + 2$.

The key point of the above theorem in constructing new self-dual codes is to supply generator matrices of self-dual codes of length 2 shorter than the length of codes we want to construct. The more we supply generator matrices of length $2n$, the bigger the chance to obtain new codes of length $2n + 2$.

Let $C$ be a self-dual code of length $2n + 2$, and let $G$ be its generator matrix. Without loss of generality we may assume that $G = (I_n | A) = (e_i | a_i)$, where $e_i$ and $a_i$ are the rows of the identity matrix $I_n$ and $A$, respectively for $1 \leq i \leq n$. Let $c$ be in $GF(q)$ such that $c^2 = -1$ in $GF(q)$. Then $C$ has also the following generator matrix

$$
G := \begin{pmatrix}
e_i - ce_2 & | & a_i - ca_2 \\
-ce_2 & | & -ca_2 \\
e_3 & | & a_3 \\
\vdots & | & \vdots \\
e_n & | & a_n
\end{pmatrix}.
$$
Deleting the first two columns and the second row of \(G\) we obtain an \((n-1)\times 2n\) matrix of the form
\[
G_0 := \begin{pmatrix}
1 & \cdots & 0 & a_i - ca_2 \\
0 & \cdots & 0 & a_i \\
I_{n-2} & & & \\
& & & a_n
\end{pmatrix}.
\]

We claim that \(G_0\) is a generator matrix of some self-dual code \(C_0\) of length \(2n\). It suffices to show that any two rows of \(G_0\) are orthogonal to each other.

The inner-product of the first row of \(G_0\) with itself equals
\[
[a_i - ca_2, a_i - ca_2] = -(c^2 + 1) = 0.
\]

For \(3 \leq i \leq n\), the inner-product of the \(i\)-th row of \(G_0\) with itself equals
\[
1 + [a_i, a_i] = 0.
\]

For \(3 \leq i \leq n\), the inner-product of the first row of \(G_0\) with the \(i\)-th row is equal to
\[
[a_i - ca_2, a_i] = [a_i, a_i] - [ca_2, a_i] = 0.
\]

For \(3 \leq i, j \leq n\), with \(i \neq j\), the inner-product of the \(i\)-th row with the \(j\)-th row is equal to
\[
0 + [a_i, a_j] = 0.
\]

Hence we have the following proposition.

**Proposition 2.2** Let \(G = (I_n \mid A) = (e_i \mid a_i)\), where \(e_i\) and \(a_i\) are the rows of the identity matrix \(I_n\) and \(A\), respectively for \(1 \leq i \leq n\), be a generator matrix of a self-dual code \(C\) of length \(2n + 2\). Then
\[
G_0 := \begin{pmatrix}
1 & \cdots & 0 & a_i - ca_2 \\
0 & \cdots & 0 & a_i \\
I_{n-2} & & & \\
& & & a_n
\end{pmatrix}
\]

is generator matrix of a self-dual code of length \(2n\).
Remark 2.3 Proposition 2.2 above is nothing but the restatement of Proposition 3.2 in [8].

2.1 Construction Algorithm

The method we use here to construct new codes is a combination of subtraction method and building-up method. Subtraction as well as building-up construction method are well known in Coding Theory. Kim’s method (Theorem 2.1) is basically a building-up method: it is possible to construct a self-dual $[2n + 2, n + 1, d + 2]$ code from a self-dual $[2n, n, \geq d]$ code. Subtraction method (Proposition 2.2) is a reverse of the building-up method: it is possible to construct a self-dual $[2n, n, \geq d]$ code from a self-dual $[2n + 2, n + 1, d + 2]$ code.

Our key step to create new codes is to supply known generator matrices $G_0$ of self-dual $[2, 2, 1, 2]$ codes as many as possible, and to use all possible vectors $x \in GF(q^2)$, for each matrix $G_0$. The algorithm is given in the Table 1 (c.f. [11]).

<table>
<thead>
<tr>
<th>Table 1</th>
<th>An algorithm to construct MDS or near-MDS self-dual codes by combination of building-up and subtraction method.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong></td>
<td>$C_{2n+2}$, a known $[2n+2, n+1, d]$ self-dual code (not necessarily (near-)MDS).</td>
</tr>
<tr>
<td><strong>Output:</strong></td>
<td>$C_{2n+2}$, the set of new $[2n+2, n+1, d]$ self-dual codes, with $d = n$ or $n + 1$.</td>
</tr>
<tr>
<td>1.</td>
<td>Construct a self-dual $[2n, n, d]$ code $C_{2n,1}$ from a given self-dual $[2n+2, n+1, d]$ code $C_{2n+2}$ by subtraction method (Proposition 2.2).</td>
</tr>
<tr>
<td>2.</td>
<td>Construct self-dual $[2n+2, n+1, d]$ codes $C_{2n+2}$ from a self-dual $[2n, n, d]$ code $C_{2n,1}$ by the building-up method (Theorem 2.1). Supply all possible values for vector $x$.</td>
</tr>
<tr>
<td>3.</td>
<td>Check the equivalence of new self-dual codes $C_{2n+2}$ from Step 2. Let say, we get $l$ inequivalent self-dual $[2n+2, n+1, d]$ codes $C_{2n+2,1}, C_{2n+2,2}, \ldots, C_{2n+2,l}$.</td>
</tr>
<tr>
<td>4.</td>
<td>For each self-dual code obtained in Step 3, return to Step 1. Denote a new self-dual $[2n, n, d]$ code by $C_{2n,2}$.</td>
</tr>
</tbody>
</table>

3 Results

In this section, we apply the above method to construct some new Hermitian self-dual MDS or near-MDS codes over $GF(9)$, $GF(25)$, and $GF(121)$. All computer calculations were done by MAGMA [12] and MATLAB.
3.1 Self-dual Near-MDS Codes Over $GF(9)$

Let $w$ be a root of a primitive polynomial $x^2 + 2x + 2 \in GF(3)[x]$ and $c := w^2$ be the element defined as in Theorem 2.1.

3.2 Length 10

Kim and Lee [8] constructed a self-dual near-MDS [10,5,5] with the following generator matrix

$$
\begin{bmatrix}
1 & 0 & w & w^2 & 1 & w & 1 & 1 & 1 & 1 \\
w & w^2 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
w^2 & w & w^2 & 1 & 0 & w & 1 & 1 & 1 & 1 \\
1 & w^2 & w^2 & w & w^2 & 1 & 0 & 1 & 1 & 1 \\
w^2 & 1 & w & w^2 & w^2 & 1 & w^4 & 1 & w \\
w & w^2 & w^2 & w^2 & w & w & w & w & w & w
\end{bmatrix}
$$

By the building-up method (Theorem 2.1) continues with the subtraction method (Proposition 2.2), we obtained three self-dual near-MDS [10,5,5] with generator matrices given below:

$$
C_{10,1} = \begin{bmatrix}
0 & 0 & 0 & 0 & w^4 & w & w^2 & w^2 & w^2 & w \\
1 & 0 & 0 & 0 & w & w^6 & w & w^4 & 0 & w^3 \\
0 & 1 & 0 & 0 & w^4 & 1 & w^3 & w^2 & w^2 & w^4 \\
0 & 0 & 1 & 0 & w^7 & w^3 & w^3 & 1 & w^6 & 0 \\
0 & 0 & 0 & 1 & w^3 & w & w & w & w & w^3
\end{bmatrix}
$$

$$
C_{10,2} = \begin{bmatrix}
0 & 0 & 0 & 0 & w^4 & w^3 & w & w^2 & w^2 & w^3 & w \\
1 & 0 & 0 & 0 & 1 & w^4 & w^6 & 0 & 1 & w^4 \\
0 & 1 & 0 & 0 & w^2 & w^5 & w^3 & w^3 & 1 & 1 \\
0 & 0 & 1 & 0 & w^3 & w & w^3 & w & w^2 & w^4 \\
0 & 0 & 0 & 1 & w^3 & w & w & w & w & w^3
\end{bmatrix}
$$

and

$$
C_{10,3} = \begin{bmatrix}
0 & 0 & 0 & 0 & w^8 & w^7 & w & w^2 & w^2 & w^3 & w \\
1 & 0 & 0 & 0 & w^2 & 0 & w^2 & w^7 & w & w^3 & w^2 \\
0 & 1 & 0 & 0 & w^8 & w^7 & w^2 & w^3 & w^4 & 0 \\
0 & 0 & 1 & 0 & w^5 & 1 & 0 & w^4 & w^7 & w^2 \\
0 & 0 & 0 & 1 & w^5 & w^7 & w^3 & w & w^2 & w^3
\end{bmatrix}
$$

Weight enumerator of the above codes is $W_{10,1}(y) = W_{10,2}(y) = 1 + 128y^5 + 1040y^6 + \cdots$, and $W_{10,3}(y) = 1 + 160y^5 + 952y^6 + \cdots$, respectively.
Since the two self-dual near-MDS [10,5,5] codes constructed by Kim and Lee [7] has weight enumerator $W(y) = 1 + 128y^3 + 1040y^6 + 4160y^7 + \cdots$ and $W(y) = 1 + 144y^5 + 960y^6 + \cdots$, respectively, then we obtained at least one new self-dual near-MDS [10,5,5] code, namely the code $C_{10,3}$.

### 3.3 Length 12

Kim and Lee [8] have constructed three self-dual near-MDS [12,6,6] codes. From the above near-MDS [10,5,5] codes, we applied the building-up method (Theorem 2.1) to construct self-dual codes of length 12. We obtained 9 self-dual near-MDS [12,6,6] codes which are not equivalent with the ones constructed by Kim and Lee [8] (see Table 2).

<table>
<thead>
<tr>
<th>No</th>
<th>Vector $x$ in Generator Matrix</th>
<th>$A_5$, $A_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(w, w, w, w, w, w, w, 0, w, w, w)$</td>
<td>480, 3456</td>
</tr>
<tr>
<td>2</td>
<td>$(w, w, w, w, w, w, w, w, w, w, w, w)$</td>
<td>480, 3456</td>
</tr>
<tr>
<td>3</td>
<td>$(w, w, w, w, w, w, w, w, w, w, w, w, w, 0, w, w, w)$</td>
<td>496, 3360</td>
</tr>
<tr>
<td>4</td>
<td>$(w, w, w, w, w, w, w, w, w, w, w, w, w, 0, w, w, w)$</td>
<td>544, 3072</td>
</tr>
<tr>
<td>5</td>
<td>$(w, w, w, w, w, w, w, w, w, w, w, w, w, 0, w, w, w)$</td>
<td>544, 3072</td>
</tr>
<tr>
<td>6</td>
<td>$(w, w, w, w, w, w, w, w, w, w, w, w, w, 0, w, w, w)$</td>
<td>624, 2592</td>
</tr>
<tr>
<td>7</td>
<td>$(w, w, w, w, w, w, w, w, w, w, w, w, w, 0, w, w, w)$</td>
<td>624, 2592</td>
</tr>
<tr>
<td>8</td>
<td>$(w, w, w, w, w, w, w, w, w, w, w, w, w, 0, w, w, w)$</td>
<td>736, 1920</td>
</tr>
</tbody>
</table>

#### 3.4 Self-dual MDS or Near-MDS Codes Over $GF(25)$

Let $w$ be a root of primitive polynomial $x^2 + 4x + 2 \in GF(25)[x]$ and $c := w^2$ be the element defined as in Theorem 2.1.

### 3.4.1 Length 10

First, the [8] provided a self-dual MDS [10,5,6] code $C_{10}'$:

$$
C_{10}' = \begin{pmatrix}
1 & 0 & 1 & 1 & 1 & 1 & 1 & w & w^3 & 0 \\
1 & 0 & w^9 & 1 & 0 & w^{22} & 1 & 1 & 1 & 1 \\
1 & 0 & w^9 & w^{19} & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & w^9 & w^{19} & w^{3} & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & w^9 & w^{19} & w^{3} & w^{7} & 1 & 0 & w^{4} & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

By subtraction method (Proposition 2.2) we obtained a self-dual [8,4] code $C_8$:
Next, by the building-up method (Theorem 2.1) we obtained 13 new (inequivalent) self-dual MDS \([10,5,6]\) codes with the same weight enumerator

\[
W(y) = 1 + 5040y^6 + 54720y^7 + 508680y^8 + 2704560y^9 + 6492624y^{10}.
\]

The (generator of) new codes are listed in the Table 3 below.

<table>
<thead>
<tr>
<th>No</th>
<th>Vector (x) in Generator Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((1,1,1,1, w, w, w))</td>
</tr>
<tr>
<td>2</td>
<td>((1,1,1,1, w^2, w^3, w^5))</td>
</tr>
<tr>
<td>3</td>
<td>((1,1,1,1, w^3, w^7, w^9))</td>
</tr>
<tr>
<td>4</td>
<td>((1,1,1,1, w^6, w^{10}, w^{12}))</td>
</tr>
<tr>
<td>5</td>
<td>((1,1,1,1, w, w, 0))</td>
</tr>
<tr>
<td>6</td>
<td>((1,1,1,1, w, 0, w))</td>
</tr>
<tr>
<td>7</td>
<td>((1,1,1,1, w^4, w^6, 0))</td>
</tr>
<tr>
<td>8</td>
<td>((1,1,1,1, w^6, w^8, w^7))</td>
</tr>
<tr>
<td>9</td>
<td>((1,1,1,1, w, w^1, 1, 0))</td>
</tr>
<tr>
<td>10</td>
<td>((1,1,1,1, w^2, w^7, w^8, w^9))</td>
</tr>
<tr>
<td>11</td>
<td>((1,1,1,1, w^4, w^6, 1, 0))</td>
</tr>
<tr>
<td>12</td>
<td>((1,1,1,1, w^6, w^2, w^8))</td>
</tr>
<tr>
<td>13</td>
<td>((1,1,1,1, w^8, w^9, w^4))</td>
</tr>
</tbody>
</table>

Moreover, we also obtained over 30 (inequivalent) near-MDS \([10,5,5]\) codes, some of them are given in Table 4 below.

<table>
<thead>
<tr>
<th>No</th>
<th>Vector (x) in Generator Matrix</th>
<th>(A_k), (A_k), (A_t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((0,0,0,0, w, w, w, w))</td>
<td>48, 4800, 55200</td>
</tr>
<tr>
<td>2</td>
<td>((0,0,0,1,1, w, 1, 0))</td>
<td>96, 4560, 55680</td>
</tr>
<tr>
<td>3</td>
<td>((0,0,0,1,1, w^1))</td>
<td>144, 4320, 56160</td>
</tr>
<tr>
<td>4</td>
<td>((0,0,0,1,1, w^2))</td>
<td>192, 4080, 56640</td>
</tr>
<tr>
<td>5</td>
<td>((0,0,0,1,1, w^3, w^2))</td>
<td>240, 3840, 57120</td>
</tr>
<tr>
<td>6</td>
<td>((0,0,0,1,1, w^4, w))</td>
<td>288, 3600, 57600</td>
</tr>
<tr>
<td>7</td>
<td>((0,0,0,1,1, w^5, w^1))</td>
<td>336, 3360, 58080</td>
</tr>
</tbody>
</table>
3.4.2 Length 12

For length 12, we obtained many (inequivalent) self-dual near-MDS codes. Some of them are listed below.

<table>
<thead>
<tr>
<th>No</th>
<th>Vector x in Generator Matrix</th>
<th>$A_0$, $A_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(w^7, w, w^5, w^2, w^4, w, w^6, w^3, w^2)$</td>
<td>456, 16272</td>
</tr>
<tr>
<td>2</td>
<td>$(1, 1, 1, 1, 1, w, w^5, w^6, w^3)$</td>
<td>480, 16128</td>
</tr>
<tr>
<td>3</td>
<td>$(w^7, w^7, w, w^5, w^2, w^4, w, w^6, w, w^3, 1)$</td>
<td>504, 15984</td>
</tr>
<tr>
<td>4</td>
<td>$(1, 1, 1, 1, 1, 1, w, w^5, w, w^3)$</td>
<td>528, 15840</td>
</tr>
<tr>
<td>5</td>
<td>$(w^7, w^7, w^7, w^5, w^2, w^4, w^2, w, w^3, 0)$</td>
<td>552, 15696</td>
</tr>
<tr>
<td>6</td>
<td>$(w^7, w^7, w^7, w^5, w^2, w^4, w^2, w, w^3, 1, 1)$</td>
<td>600, 15408</td>
</tr>
<tr>
<td>7</td>
<td>$(w^7, w^7, w^7, w^5, w^2, w^4, w^2, w, w^3, 0, w^2)$</td>
<td>624, 15264</td>
</tr>
<tr>
<td>8</td>
<td>$(w^7, w^7, w^7, w^5, w^2, w^4, w^2, w, w^3)$</td>
<td>648, 15120</td>
</tr>
<tr>
<td>9</td>
<td>$(w^7, w^7, w^7, w^5, w^2, w^4, w^2, w, w^3)$</td>
<td>672, 14976</td>
</tr>
<tr>
<td>10</td>
<td>$(w^7, w^7, w^7, w^5, w^2, w^4, w^2, w, w^3)$</td>
<td>696, 14852</td>
</tr>
<tr>
<td>11</td>
<td>$(1, 1, 1, 1, 1, 1, 1, w, w^5, w, w^3, 0)$</td>
<td>720, 14688</td>
</tr>
<tr>
<td>12</td>
<td>$(w^7, w^7, w^7, w^5, w^2, w^4, w^2, w, w^3)$</td>
<td>744, 14544</td>
</tr>
<tr>
<td>13</td>
<td>$(w^7, w^7, w^7, w^5, w^2, w^4, w^2, w, w^3)$</td>
<td>768, 14400</td>
</tr>
<tr>
<td>14</td>
<td>$(1, 1, 1, 1, 1, 1, 1, w, w^5, w^2)$</td>
<td>792, 14256</td>
</tr>
<tr>
<td>15</td>
<td>$(w^7, w^7, w^7, w^5, w^2, w^4, w^2, w, w^3)$</td>
<td>816, 14112</td>
</tr>
<tr>
<td>16</td>
<td>$(1, 1, 1, 1, 1, 1, 1, w, w^5, w, w^3)$</td>
<td>840, 13968</td>
</tr>
<tr>
<td>17</td>
<td>$(1, 1, 1, 1, 1, 1, 1, w, w^5)$</td>
<td>864, 13824</td>
</tr>
<tr>
<td>18</td>
<td>$(1, 1, 1, 1, 1, 1, 1, w, w^5, w)$</td>
<td>888, 13680</td>
</tr>
<tr>
<td>19</td>
<td>$(1, 1, 1, 1, 1, 1, 1, w, w^5, 0)$</td>
<td>912, 13536</td>
</tr>
<tr>
<td>20</td>
<td>$(w^7, w^7, w^7, w^5, w^2, w^4, w^2, w, w^3)$</td>
<td>936, 13392</td>
</tr>
<tr>
<td>21</td>
<td>$(w^7, w^7, w^7, w^5, w^2, w^4, w^2, w, w^3)$</td>
<td>960, 13248</td>
</tr>
<tr>
<td>22</td>
<td>$(1, 1, 1, 1, 1, 1, w, w^5, w, w^3, 0)$</td>
<td>984, 13004</td>
</tr>
<tr>
<td>23</td>
<td>$(w^7, w^7, w^7, w^5, w^2, w^4, w^2, w, w^3)$</td>
<td>1004, 12864</td>
</tr>
<tr>
<td>24</td>
<td>$(w^7, w^7, w^7, w^5, w^2, w^4, w^2, w, w^3, 0, w^2)$</td>
<td>1028, 12720</td>
</tr>
<tr>
<td>25</td>
<td>$(w^7, w^7, w^7, w^5, w^2, w^4, w^2, w, w^3)$</td>
<td>1032, 12584</td>
</tr>
<tr>
<td>26</td>
<td>$(w^7, w^7, w^7, w^5, w^2, w^4, w^2, w, w^3)$</td>
<td>1056, 12440</td>
</tr>
<tr>
<td>27</td>
<td>$(1, 1, 1, 1, 1, 1, 1, w, w^5, w, w^3, 0)$</td>
<td>1080, 12296</td>
</tr>
<tr>
<td>28</td>
<td>$(1, 1, 1, 1, 1, 1, 1, w, w^5, 0)$</td>
<td>1104, 12152</td>
</tr>
<tr>
<td>29</td>
<td>$(1, 1, 1, 1, 1, 1, 1, w, w^5, w^2)$</td>
<td>1128, 11908</td>
</tr>
<tr>
<td>30</td>
<td>$(1, 1, 1, 1, 1, 1, 1, w, w^5, w^3)$</td>
<td>1152, 11764</td>
</tr>
</tbody>
</table>

3.4.3 Length 14

Again, from self-dual codes of length 12, by the building-up method, we obtained over 20 (inequivalent) self-dual near-MDS [14,7,7] codes. The codes as well as their weight enumerators are listed below.
Let \( w \) be a root of primitive polynomial \( x^2 + 5x + 2 \in GF(121)[x] \) and \( c := w^2 \) be the element defined in Theorem 2.1.

### 3.5 Self-dual MDS or Near-MDS Codes Over \( GF(121) \)

#### 3.5.1 Length 4

From a self-dual code \((1 \, w^5)\) of length 2, by the building-up method, we obtained a self-dual MDS \([4,2,3]\) code

\[
\begin{pmatrix}
1 & 0 & 1 & w^6 \\
 w^3 & w^6 & 1 & w^5
\end{pmatrix}
\]

having weight enumerator \(1 + 480y^3 + 14160y^4\). We also obtained a self-dual near-MDS \([4,2,2]\) code
Djoko Suprijanto, et al.

\[
\begin{bmatrix}
1 & 0 & 0 & w^3 \\
1 & w^{65} & 1 & w^5
\end{bmatrix}
\]

having weight enumerator \(1 + 240y^3 + 14400y^4\).

### 3.5.2 Length 6

From the above MDS code, again by the building-up method, we obtained three (inequivalent) self-dual MDS [6,3,4] codes with the same weight enumerator

\[1 + 1800y^4 + 84240y^5 + 1685520y^6.\]

**Table 7** Self-dual MDS [6,3,4] codes over \(GF(121)\).

<table>
<thead>
<tr>
<th>No</th>
<th>Vector x in Generator Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((0,1,1,w^3))</td>
</tr>
<tr>
<td>2</td>
<td>((0,1,1,w^{65}))</td>
</tr>
<tr>
<td>3</td>
<td>((0,1,1,w^{30}))</td>
</tr>
</tbody>
</table>

We also obtained several (inequivalent) self-dual near-MDS [6,3,3] codes as given below.

**Table 8** Self-dual near-MDS [6,3,3] codes over \(GF(121)\).

<table>
<thead>
<tr>
<th>No</th>
<th>Vector x in Generator Matrix</th>
<th>(A_1, A_2, A_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((0,0,1,w^6))</td>
<td>120, 1440, 84600, 1685400</td>
</tr>
<tr>
<td>2</td>
<td>((0,0,1,w^{65}))</td>
<td>240, 1080, 84960, 168580</td>
</tr>
<tr>
<td>3</td>
<td>((0,0,1,w,1))</td>
<td>480, 14880, 56640, 1699560</td>
</tr>
<tr>
<td>4</td>
<td>((0,1,1,w^{6}))</td>
<td>600, 14520, 57000, 1699440</td>
</tr>
</tbody>
</table>

### 3.5.3 Length 8

Again, from self-dual codes of length 6, by the building-up method, we obtained a self-dual MDS [8,4,5] code

\[
\begin{bmatrix}
1 & 0 & w^9 & w^9 & w^9 & w^9 & w^9 & w^{11} \\
w^{37} & w^{102} & 1 & 0 & 0 & 1 & 1 & w^3 \\
w^7 & w^{72} & w^{69} & w^{14} & 1 & 0 & 1 & w^6 \\
w^{69} & w^{14} & w^{60} & w^5 & w^{13} & w^{98} & 1 & w^5
\end{bmatrix}
\]

having weight enumerator

\[W(y) = 1 + 6720y^5 + 389760y^6 + 13372800y^7 + 200589600y^8.\]

There are also several (inequivalent) self-dual near-MDS [8,4,4] codes as given below.
Table 9  Self-dual near-MDS [8,4,4] codes over $GF(121)$.

<table>
<thead>
<tr>
<th>No</th>
<th>Vector $x$ in Generator Matrix</th>
<th>$A_x$, $A_y$, $A_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(w^7, w^3, w^5, w^9, w^{11})$</td>
<td>240, 5760, 391200</td>
</tr>
<tr>
<td>2</td>
<td>$(w^7, w^3, w^5, w^9, w^{13}, w^{17})$</td>
<td>480, 4800, 392640</td>
</tr>
<tr>
<td>3</td>
<td>$(w^7, w^3, w^5, w^9, w^{13}, w^{11})$</td>
<td>720, 3840, 394080</td>
</tr>
<tr>
<td>4</td>
<td>$(w^7, w^3, w^5, w^9, w^{13}, w^{11})$</td>
<td>960, 2880, 395520</td>
</tr>
<tr>
<td>5</td>
<td>$(w^7, w^3, w^5, w^9, w^{13}, w^{11})$</td>
<td>1200, 1920, 396960</td>
</tr>
</tbody>
</table>

3.5.4 Length 10

From self-dual codes of length 8, by the building-up method, we obtained a self-dual MDS [10,5,6] code

\[
\begin{pmatrix}
1 & 0 & w^{29} & w^{29} & w^{29} & w^{29} & w^{34} & w^{100} & w^{97} \\
0 & 0 & w^{14} & w^{14} & w^{9} & w^{9} & w^{9} & w^{9} & w^{31} \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & w^{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w^{5} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w^{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w^{7} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w^{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w^{9} \\
\end{pmatrix}
\]

with weight enumerator

\[W(y) = 1 + 25200y^6 + 1656000y^7 + 74601000y^8 + \ldots.\]

There are also several (inequivalent) self-dual near-MDS codes as given below.

Table 10  Self-dual near-MDS [10,5,5] codes over $GF(121)$.

<table>
<thead>
<tr>
<th>No</th>
<th>Vector $x$ in Generator Matrix</th>
<th>$A_x$, $A_y$, $A_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(w^7, w^3, w^5, w^9, w^{11}, w^{13}, w^{17})$</td>
<td>240, 24000, 1658400</td>
</tr>
<tr>
<td>2</td>
<td>$(w^7, w^3, w^5, w^9, w^{11}, w^{13}, w^{17})$</td>
<td>480, 22800, 1660800</td>
</tr>
<tr>
<td>3</td>
<td>$(w^7, w^3, w^5, w^9, w^{11}, w^{13}, w^{17})$</td>
<td>720, 21600, 1663200</td>
</tr>
<tr>
<td>4</td>
<td>$(w^7, w^3, w^5, w^9, w^{11}, w^{13}, w^{17})$</td>
<td>960, 20400, 1665600</td>
</tr>
<tr>
<td>5</td>
<td>$(w^7, w^3, w^5, w^9, w^{11}, w^{13}, w^{17})$</td>
<td>1200, 19200, 1668000</td>
</tr>
<tr>
<td>6</td>
<td>$(w^7, w^3, w^5, w^9, w^{11}, w^{13}, w^{17})$</td>
<td>1440, 18000, 16704000</td>
</tr>
</tbody>
</table>

4  Remark

Let $C$ and $C'$ be two linear $[n,k,d]$ codes which have weight distributions $(A_1, A_2, \ldots, A_n)$ and $(A'_1, A'_2, \ldots, A'_n)$, respectively. It is also well known (see [13]) that from viewpoint of decoding error probability, the code $C$ performs better than $C'$ if $(A_1, A_2, \ldots, A_n) \prec (A'_1, A'_2, \ldots, A'_n)$, where $\prec$ means
lexicographical ordering. In the above tables, we short the MDS or near-MDS codes due to their performance with respect to decoding error probability. Moreover, recently Buyuklieva, et al. [14] proved that in binary case self-dual codes perform better than non self-dual codes, for the codes with the same parameters. It is interesting to know whether the similar situation happens for the non-binary case, in particular in the case of Euclidean self-dual or Hermitian self-dual (near-) MDS codes, etc. This observation, which is now in preparation, will be published elsewhere in a separate paper.

5 Conclusion

As mentioned above there are many self-dual (near-) MDS codes over GF(9), GF(25), and GF(121) of several small lengths constructed by the building-up method as well as our simple algorithm, which combine building-up and subtraction method. To our best knowledge it was unnoticed before in any scientific publication. We concern also with self-dual near-MDS codes because of two reasons: (1) From perspective of capability of error-correcting codes, it is well-known fact that self-dual MDS and self-dual near-MDS are not very different; (2) From cryptographic application, in particular in secret sharing schemes, self-dual near-MDS instead of self-dual MDS codes are important (see, e.g., [11],[12]). There is some expectation to obtain many more self-dual MDS or near-MDS codes over these fields. It will be very good if someone can provide complete classifications of such codes.

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[4] Dodunekov, S., Applications of Near-MDS Codes in Cryptography, in Enhancing Cryptographic Primitive with Techniques from Error-


