Boolean Algebra of C-Algebras

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Abstract. A C-algebra is the algebraic form of the 3-valued conditional logic, which was introduced by F. Guzman and C.C. Squier in 1990. In this paper, some equivalent conditions for a C-algebra to become a boolean algebra in terms of congruences are given. It is proved that the set of all central elements $B(A)$ is isomorphic to the Boolean algebra $\mathcal{B}_{S(A)}$ of all C-algebras $S_a$, where $a \in B(A)$. It is also proved that $B(A)$ is isomorphic to the Boolean algebra $\mathcal{B}_{R(A)}$ of all C-algebras $A_a$, where $a \in B(A)$.

Keywords: Boolean algebra; C-algebra; central element; permutable congruences.

1 Introduction

The concept of C-algebra was introduced by Guzman and Squier as the variety generated by the 3-element algebra $C=\{T,F,U\}$. They proved that the only subdirectly irreducible C-algebras are either $C$ or the 2-element Boolean algebra $B = \{T,F\}$ [1,2].

For any universal algebra $A$, the set of all congruences on $A$ (denoted by $\text{Con } A$) is a lattice with respect to set inclusion. We say that the congruences $\theta, \phi$ are permutable if $\theta \circ \phi = \phi \circ \theta$. We say that $\text{Con } A$ is permutable if $\theta \circ \phi = \phi \circ \theta$ for all $\theta, \phi \in \text{Con } A$. It is known that $\text{Con } A$ need not be permutable for any C-algebra $A$.

In this paper, we give sufficient conditions for congruences on a C-algebra $A$ to be permutable. Also we derive necessary and sufficient conditions for a C-algebra $A$ to become a Boolean algebra in terms of the congruences on $A$. We also prove that the three Boolean algebras $B(A)$, the set of C-algebras $\mathcal{B}_{S(A)}$ and the set of C-algebras $\mathcal{B}_{R(A)}$ are isomorphic to each other.
2 Preliminaries

In this section we recall the definition of a C-algebra and some results from [1,3,5] which will be required later.

Definition 2.1. By a C-algebra we mean an algebra \( < A, \wedge, \vee, ' > \) of type (2,2,1) satisfying the following identities [1].

(a) \( x'' = x \);
(b) \( (x \wedge y)' = x' \vee y' \);
(c) \( x \wedge (y \wedge z) = (x \wedge y) \wedge z \);
(d) \( x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \);
(e) \( (x \vee y) \wedge z = (x \wedge z) \vee (x' \wedge y \wedge z) \);
(f) \( x \vee (x \wedge y) = x \);
(g) \( (x \wedge y) \vee (y \wedge x) = (y \wedge x) \vee (x \wedge y) \).

Example 2.2. [1]:

The 3-element algebra \( C = \{ T, F, U \} \) is a C-algebra with the operations \( \wedge, \vee \) and \( ' \) defined as in the following tables.

\[
\begin{array}{ccc}
| x | x' | T | F | U | T | F | U | T | F | U |
\hline
T & F & T & T & F & U & T & T & T & T \\
F & T & F & F & F & F & F & T & F & U \\
U & U & U & U & U & U & U & U & U & U \\
\end{array}
\]

Every Boolean algebra is a C-algebra.

Lemma 2.3. Every C-algebra satisfies the following laws [1,3,5].

(a) \( x \wedge x = x \);
(b) \( x \wedge y = x \wedge (x' \vee y) = (x' \vee y) \wedge x \);
(c) \( x \vee (x' \wedge x) = x \);
(d) \( (x \vee x') \wedge y = (x \wedge y) \vee (x' \wedge y) \);
(e) \( (x \vee x') \wedge x = x \);
(f) \( x \vee x' = x' \vee x \);
(g) \( x \vee y \vee x = x \vee y \);
(h) \( x \wedge x' \wedge y = x \wedge x' \);
(i) \( x \wedge (y \vee x) = (x \wedge y) \vee x \).

The duals of all above statements are also true.
Definition 2.4. If $A$ has identity $T$ for $\land$ (that is, $x \land T = T \land x = x$ for all $x \in A$), then it is unique and in this case, we say that $A$ is a $C$-algebra with $T$. We denote $T'$ by $F$ and this $F$ is the identity for $\lor$ [1].

Lemma 2.5 [1]: Let $A$ be a $C$-algebra with $T$ and $x, y \in A$. Then

(i) $x \lor y = F$ if and only if $x = y = F$
(ii) if $x \lor y = T$ then $x \lor x' = T$
(iii) $x \lor T' = x \lor x'$
(iv) $x \land F = x \land x'$

Theorem 2.6. Let $A, \land, \land, \lor, \lor$ be a $C$-algebra. Then the following are equivalent [6]:

(i) $A$ is a Boolean algebra.
(ii) $x \lor (y \land x) = x$, for all $x, y \in A$.
(iii) $x \land y = y \land x$, for all $x, y \in A$.
(iv) $(x \lor y) \land y = y$, for all $x, y \in A$.
(v) $x \lor x'$ is the identity for $\land$, for every $x \in A$.
(vi) $x \lor x' = y \lor y'$, for all $x, y \in A$.
(vii) $A$ has a right zero for $\land$.
(viii) for any $x, y \in A$, there exists $a \in A$ such that $x \lor a = y \lor a = a$.
(ix) for any $x, y \in A$, if $x \lor y = y$, then $y \land x = x$.

Definition 2.7. Let $A$ be a $C$-algebra with $T$. An element $x \in A$ is called a central element of $A$ if $x \land x' = T$. The set of all central elements of $A$ is called the Centre of $A$ and is denoted by $B(A)$ [5].

Theorem 2.8. Let $A$ be a $C$-algebra with $T$. Then $A, \land, \land, \lor, \lor$ is a Boolean Algebra [5].

3 Some Properties of a $C$-algebra and Its Congruences

In this section we prove some important properties of a $C$-algebra and we give sufficient conditions for two congruences on a $C$-algebra $A$ to be permutable. Also we derive necessary and sufficient conditions for a $C$-algebra $A$ to become a Boolean algebra in terms of the congruences on $A$.

Lemma 3.1. Every $C$-algebra satisfies the following identities:
Boolean Algebra of C-Algebras

(i) \( x \vee y = x \vee (y \land x') \);
(ii) \( x \land y = x \land (y \lor x') \).

**Proof.** Let \( A \) be a C-algebra and \( x, y \in A \). Now,
\[
\begin{align*}
x \vee y &= x \vee (x' \land y) = x \vee (x' \land y \land x') = [x \land (x \lor x')] \lor [x' \land y \land (x \lor x')] = [x \land (x \lor x')] \lor [x' \land y \land (x \lor x')] = (x \lor y) \\
\land (x \lor x') &= x \lor (y \land x').
\end{align*}
\]
Similarly \( x \land y = x \land (y \lor x') \).

**Lemma 3.2.** Let \( A \) be a C-algebra and \( x, y \in A \). Then \( x \lor y \lor x' = x \lor y \lor y' \).

**Proof.** Let \( A \) be a C-algebra and \( x, y \in A \). By Lemma 2.3[b],[f] and Lemma 3.1, we have \( x \lor y \lor x' = x \lor ((y' \land x') \lor y) = [x \lor (y' \land x')] \lor y = (x \lor y') \lor y = x \lor y \lor y' \).

**Lemma 3.3.** Let \( A \) be a C-algebra with \( T, x, y \in A \) and \( x \land y = F \). Then \( x \lor y = y \lor x \).

**Proof.** Suppose that \( x \land y = F \). Then \( F = x \land y = x \land (x' \lor y) = (x \land x') \lor (x \land y) = (x \land x') \lor F = x \land x' \). Now \( x \lor y = F \lor (x \lor y) = (x \lor y) \lor (x \lor y) = (x \lor x) \lor (x' \lor y \lor x \lor y) \) (By Def 2.1) \( = (x \lor y) \land (x' \lor y \land y) \) \( = (x \lor y) \lor (x' \lor y \lor x) = F \lor (y \land x) = y \lor x \).

In [6], it is proved that if \( A \) is a C-algebra with \( T \) then \( B(A) = \{ a \in A \mid a \lor a' = T \} \) is a Boolean algebra under the same operations \( \land, \lor, ' \) in the C-algebra \( A \). Now we prove the following.

**Theorem 3.4.** Let \( A \) be a C-algebra with \( T \) and \( a, b \in A \) such that \( a \lor b \in B(A) \). Then \( a \in B(A) \).

**Proof.** Let \( A \) be a C-algebra with \( T \) and \( a, b \in A \) such that \( a \lor b \in B(A) \). Then \( T = (a \lor b) \lor (a \lor b') = (a \lor b) \lor (a' \land b') = (a \lor b \lor a') \land (a \lor b \lor b') \) (By Theorem 3.2) \( = a \lor b \lor a' \).
Therefore, \( T = a \lor b \lor a' \) ...(I)
Now, \( a \lor a' = (a \lor a') \land T \)
\[ = (a \lor a') \land (a \lor b \lor a') \quad \text{(by I)} \]
\[ = (a \land (a \lor b \lor a')) \lor (a' \land (a \lor b \lor a')) \]
\[ = a \lor (a' \land (a \lor b \lor a')) \]
\[ = a \lor (a \lor b \lor a') \quad \text{(By 2.3[b])} \]
\[ = a \lor b \lor a' = T. \]

Hence \( a \in B(A) \).

The converse of the above theorem need not be true. For example, in the C-algebra \( C, F \in B(C) \) but \( F \lor U = U \not\in B(C) \). We have the following consequence of the above theorem.

**Corollary 3.5.** Let A be a C-algebra with \( T, a, b \in A \) and \( a \land b \in B(A) \). Then \( a \in B(A) \).

**Proof.** Let \( a \lor b \in B(A) \). Then we have, \( (a \land b)' \in B(A) \Rightarrow a' \lor b' \in B(A) \Rightarrow a' \in B(A) \Rightarrow a \in B(A) \).

In [1], it is proved that if A is a C-algebra, then \( \theta_x = \{(p, q) | x \land p = x \land q\} \) is a congruence on A and \( \theta_x \cap \theta_y = \theta_{x \lor y} \). In [6], if A is C-algebra with T then \( \theta_x \) is a factor congruence if and only if \( x \in B(A) \). They also proved that \( \theta_x, \theta_y \) are permutable congruences whenever both \( x, y \in B(A) \). Now we prove some important properties of these congruences.

**Theorem 3.6.** Let A be a C-algebra with T and \( a, b \in A \). Then we have the following (i) \( \theta_{a \land b} = \theta_{b \lor a} \); (ii) \( \theta_a \circ \theta_b \subseteq \theta_{a \land b} \).

**Proof.** (i) \( (x, y) \in \theta_{a \land b} \)
\[ \Rightarrow a \land b \land x = a \land b \land y \]
\[ \Rightarrow b \land a \land b \land x = b \land a \land b \land x \]
\[ \Rightarrow b \land a \land x = b \land a \land x \]
\[ \Rightarrow (x, y) \in \theta_{b \lor a} \]
Therefore \( \theta_{a \land b} \subseteq \theta_{b \lor a} \). Similarly, \( \theta_{b \lor a} \subseteq \theta_{a \land b} \). Hence \( \theta_{a \land b} = \theta_{b \lor a} \).
(ii) Let \((x, y) \in \theta_a \circ \theta_b\). Then there exists \(z \in A\) such that \((x, z) \in \theta_b\) and \((z, y) \in \theta_a\). Thus \(b \land x = b \land z\) and \(a \land z = a \land y\). Now, \(a \land b \land x = a \land b \land z = a \land b \land a \land z = a \land b \land a \land y = a \land b \land y\). Therefore, \((x, y) \in \theta_{a \land b}\). Thus \(\theta_a \circ \theta_b \subseteq \theta_{a \land b}\).

In the following we give an example of a C-algebra \(G\) without \(T\) in which the \(\text{Con } A\) is not permute.

**Example 3.7.** Consider the C-algebra \(G = \{a_1, a_2, a_3, a_4, a_5\}\) where \(a_1 = (T, U)\), \(a_2 = (F, U)\), \(a_3 = (U, T)\), \(a_4 = (U, F)\), \(a_5 = (U, U)\) under pointwise operations in \(C\).

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This algebra \((G, \lor, \land, ')\) is a C-algebra with out \(T\). Let \(\Delta = \text{diagonal of } A\). Then we have the following:

\[
\theta_{a_1} = \{(x, y) | a_1 \land x = a_1 \land y \}
\]
\[
= \Delta \cup \{(a_3, a_4), (a_4, a_5), (a_5, a_3), (a_3, a_5)(a_5, a_3), (a_3, a_5)\}
\]
\[
\theta_{a_3} = \Delta \cup \{(a_1, a_2), (a_2, a_3), (a_3, a_1), (a_1, a_1)(a_5, a_2), (a_1, a_5)\}
\]
Now, \( \theta_{a_1} \circ \theta_{a_3} = \Delta \cup \theta_{a_1} \cup \theta_{a_3} \cup \{(a_4, a_1), (a_4, a_2)(a_3, a_1), (a_3, a_2)\} \)
\( \theta_{a_3} \circ \theta_{a_1} = \Delta \cup \theta_{a_3} \cup \theta_{a_1} \cup \{(a_2, a_4), (a_2, a_5)(a_1, a_3), (a_1, a_4)\} \)
Therefore \( \theta_{a_1} \circ \theta_{a_3} \neq \theta_{a_3} \circ \theta_{a_1} \).

**Theorem 3.8.** Let A a C-algebra with T and \( a \in B(A) \). Then for any \( b \in A, \theta_{a} \circ \theta_{b} \) permute and \( \theta_{a} \circ \theta_{b} = \theta_{a,b} \).

**Proof.** Let A be a C-algebra with T and \( a \in B(A) \). By Theorem 3.6, \( \theta_{a} \circ \theta_{b} \subseteq \theta_{a,b} \). Now let \( (p, q) \in \theta_{a,b} \). Then \( a \wedge b \wedge p = a \wedge b \wedge q \Rightarrow b \wedge a \wedge p = b \wedge a \wedge q \). Consider, \( r = (a \wedge p) \vee (a' \wedge q) \). Now \( a \wedge r = a \wedge [(a \wedge p) \vee (a' \wedge q)] = (a \wedge p) \vee (a \wedge a' \wedge q) = (a \wedge p) \vee (F \wedge q) = (a \wedge p) \vee F = a \wedge p \). Therefore \( (r, p) \in \theta_{a} \Rightarrow (p, r) \in \theta_{a} \). Now, \( b \wedge r = b \wedge [(a \wedge p) \vee (a' \wedge q)] = [b \wedge a \wedge p] \vee [b \wedge a' \wedge q] = (b \wedge a \wedge q) \vee (b \wedge a' \wedge q) = b \wedge ((a \wedge q) \vee (a' \wedge q)) = b \wedge (a \wedge q) \) (since \( a \in B(A) \)).

**Corollary 3.9.** Let A be a C-algebra with T and \( a, b \in A \). Then i) \( a \vee b \in B(A) \Rightarrow \theta_{a} \circ \theta_{b} = \theta_{a,b} \); ii) \( a \wedge b \in B(A) \Rightarrow \theta_{a} \circ \theta_{b} = \theta_{a,b} \).

**Proof.** i) We know that if \( a \vee b \in B(A) \) then \( a \in B(A) \) and hence by the above theorem \( \theta_{a} \circ \theta_{b} = \theta_{b} \circ \theta_{a} = \theta_{a,b} \). Similarly, we can prove ii).

Let A be a C-algebra. If Con(A) is permutable, then A need not be a Boolean algebra. For example, in the C-algebra C, the only congruences are \( \Delta, \vee \) and they are permutable. But C is not a Boolean algebra. Now we give equivalent conditions for a C-algebra to become a Boolean algebra in terms of congruence relations.

**Theorem 3.10.** Let \((A, \vee, \wedge, \sim)\) be a C-algebra with T. Then the following are equivalent. (i) Let \((A, \vee, \wedge, \sim)\) be a Boolean algebra. (ii) \( \theta_{a} \cap \theta_{x} \cap \theta_{x} = \Delta \) for all \( x \in A \). (iii) \( \theta_{x} \vee \theta_{x} = \Delta \) for all \( x \in A \).
Proof. (1) ⇒ (2): Let $A$ be a Boolean algebra and $x \in A$. Let $(p, q) \in \Theta_x \cap \Theta_x'$. Then $x \land p = x \land q$ and $x' \land p = x' \land q$. Now, $p = (x \lor x') \land p = (x \land p) \lor (x' \land q) = (x \land q) \lor (x' \land q) = (x \lor x') \land q = q$. Thus $\Theta_x \cap \Theta_x' \subseteq \Delta$. Therefore $\Theta_x \cap \Theta_x' = \Delta$. Since $\Theta_x \cap \Theta_x' = \Theta_{x \lor x'}$, we get (ii) ⇒ (iii). (iii) ⇒ (i): Suppose $\Theta_x \circ \Theta_x = A \times A$. Let $(p, q) \in A \times A$. Write $t = (x \land p) \lor (x' \land q)$. Now, $x \land t = x \land ((x \land p) \lor (x' \land q)) = (x \land p) \lor (x' \land q) = (x \lor x') \land q = x \land p$. Also, $x' \land t = x' \land ((x \land p) \lor (x' \land q)) = (x' \land x) \lor (x' \land q) = (x' \land p) \lor (x' \land q) = t$. Therefore $(p, t) \in \Theta_x$ and $(t, q) \in \Theta_x'$. Thus $(p, q) \in \Theta_x \circ \Theta_x$. Hence we get $\Theta_x \circ \Theta_x = A \times A$. Also $\Theta_x \cap \Theta_x' = \Theta_{x \lor x'} = \Delta$. That is $\Theta_x$ and $\Theta_x'$ are permutable factor congruences. Therefore, by Theorem 2.6, we have $x \in B(A)$. Thus $A = B(A)$ and hence $A$ is a Boolean algebra.

4 The C-algebra $S_x$

We prove that, for each $x \in A$, $S_x = \{x \lor t \mid t \in A\}$ is itself a C-algebra under induced operations $\land, \lor$ and the unary operation is defined by $(x \lor t)^* = x \land t'$. We observe that $S_x$ need not be a subalgebra of $A$ because the unary operation in $S_x$ is not the restriction of the unary operation on $A$. Also for each $x \in A$, the set $A_x = \{x \lor t \mid t \in A\}$ is a C-algebra in which the unary operation is given by $(x \lor t)^* = x \land t'$. We prove that the $B(A)$ is isomorphic to the Boolean algebra $B_{S(A)}$ of all C-algebras $S_x$ where $a \in B(A)$. Also, we prove that $B(A)$ is isomorphic to the Boolean algebra $B_{R(A)}$ of all C-algebras $A_a, a \in B(A)$.

Theorem 4.1. Let $< A, \land, \lor, \ast >$ be a C-algebra, $x \in A$ and $S_x = \{x \lor t \mid t \in A\}$. Then $< S_x, \land, \lor, \ast >$ is a C-algebra with $x$ as the identity for $\lor$, where $\land$ and $\lor$ are the operations in $A$ restricted to $S_x$ and for any $x \lor t \in S_x$, here $(x \lor t)^*$ is $x \lor t'$.

Proof. Let $t, r, s \in A$. Then $(x \lor t) \lor (x \lor r) = x \lor (t \lor r) \in S_x$ and $(x \lor t) \land (x \lor r) = x \land (t \lor r) \in S_x$. Thus $\land, \lor$ are closed in $S_x$. Also $^*$ is closed in $S_x$. Consider $(x \lor t)^* = x \lor (x' \land t) = x \lor t$. Now $[(x \land t) \land (x \lor r)]^*$
\[ x \vee (t \wedge r) = (x \vee t') \wedge (x \vee r') = (x \vee t') \vee (x \vee r') = (x \vee t) \vee (x \vee r). \]

Now, consider

\[ (x \vee t) \wedge (x \vee r) = (t \wedge r) \quad \text{and} \quad (x \vee t) \wedge (x \vee r) = (t \wedge r). \]

The remaining identities of a C-algebra also hold in \( S \), because they hold in \( A \). Hence, \( S \) is itself a C-algebra.

\[ x \text{ is the identity for } v \text{ because } x \vee x = x \vee x. \]

Theorem 4.2. Let \( A \) be a C-algebra. Then the following holds.

(1) \( S_x = S_y \) if and only if \( x = y \);

(2) \( S_x \cap S_y \subseteq S_{x \wedge y} \);

(3) \( S_x \cap S_\vee = S_{x \vee y} \);

(4) \( (S_x)_{x \wedge y} = S_{x \wedge y} \).

Proof. (1) Suppose \( S_x = S_y \). Since \( x = x \vee x \in S_x = S_y \) and \( y = y \vee y \in S_y = S_x \), therefore \( x = y \vee t \) and \( y = x \vee r \) for some \( t, r \in A \). Now, \( x = y \vee t = (y \vee t) \wedge (y \vee y \vee t) \) and \( y = x \vee r = (x \vee r) \wedge (x \vee x \vee r) \). Thus, \( x \vee t = x \vee y \vee r \in S_{x \wedge y} \).

(2) Suppose \( t \in S_x \cap S_y \). Then \( t = x \vee s = y \vee r \) for some \( s, r \in A \). Now, \( t = x \vee x \vee s = x \vee t = x \vee y \vee r = S_{x \wedge y} \). Hence \( S_x \cap S_\vee = S_{x \vee y} \).

(3) \( (S_x)_{x \wedge y} \subseteq S_{x \wedge y} \) by (2). Since \( x \vee t' = x' \vee x \) we have \( S_{x \vee y} \subseteq S_x \cap S_\vee \). Hence \( S_x \cap S_\vee = S_{x \vee y} \).

(4) \( (S_x)_{x \wedge y} = S_{x \wedge y} \).

Theorem 4.3. Let \( A \) be a C-algebra with \( x \in A \), then the mapping \( \alpha_x : A \to S_x \) defined by \( \alpha_x(t) = x \vee t \) for all \( t \in A \) is a homomorphism of \( A \) to \( S_x \) with kernel \( \ker \alpha_x \) and hence \( A/\ker \alpha_x \cong S_x \).

Proof. Let \( t, r \in A \). Then \( \alpha_x(t \wedge r) = x \vee t \wedge r = x \vee t \wedge x \vee r = \alpha_x(t) \wedge \alpha_x(r) \) and \( \alpha_x(t') = x \vee t' = (x \vee t) \vee = (\alpha_x(t)) \vee \). Clearly, \( \alpha_x(t \vee r) = \alpha_x(t) \vee \alpha_x(r) \).

Also \( \alpha_x(T) = x \vee T = x \vee x' \), which is the identity for \( \wedge \) in \( S_x \). Therefore \( \alpha_x \) is a homomorphism. Hence by the fundamental theorem of homomorphism \( A/\ker \alpha_x \cong S_x \) and \( \ker \alpha_x = \{(t, r) \mid (t, r) \in A \times A \} \)
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\[ x \lor t = x \lor r = \{(t, r) \in A \times A \mid x' \land t = x' \land r\} \theta_c, \quad \text{(by Lemma 2.3 [b])} \]

and hence \( A/\theta_c \cong S_c \).

**Theorem 4.4.** Let \( A \) be a C-algebra with \( T \) and \( a \in B(A) \), then \( A \cong S_a \times S_d \).

**Proof.** Define \( \alpha : A \to S_a \times S_d \) by \( \alpha(x) = (\alpha_a(x), \alpha_d(x)) \) for all \( x \in A \). Then, by Theorem 4.3, \( \alpha \) is well-defined and \( \alpha \) is a homomorphism. Now, we prove that \( \alpha \) is one-one. Let \( x, y \in A \). Then \( \alpha(x) = \alpha(y) \Rightarrow (\alpha_a(x), \alpha_d(x)) = (\alpha_a(y), \alpha_d(y)) \Rightarrow (a \lor x, a' \lor x) = (a \lor y, a' \lor y) \Rightarrow a \lor x = a \lor y \) and \( a' \lor x = a' \lor y \). Now, \( x = F \lor x = (a \land a') \lor x = (a \lor x) \land (a' \lor x) = (a \lor y) \land (a' \lor y) = y \). Finally, we prove that \( \alpha \) is onto. Let \( (x, y) \in S_a \times S_d \). Then \( x = a \lor t \) and \( y = a' \lor r \) for some \( t, r \in A \). Therefore, \( a \lor x = x \)

\[ a \lor y = a \lor a' \lor y = T \lor y = T \quad \text{and} \quad a' \lor x = T, a' \lor y = y. \]

Now, \( \alpha(x \lor y) = (\alpha_a(x \lor y), \alpha_d(x \lor y)) \)

\[ = (a \lor (x \lor y), a' \lor (x \lor y)) \]

\[ = ((a \lor x) \land (a \lor y), (a' \lor x) \land (a' \lor y)) \]

\[ = (x \lor T, T \lor y) \]

\[ = (x, y). \]

Therefore, \( \alpha \) is onto and hence \( \alpha \) is an isomorphism. Therefore \( A \cong S_a \times S_d \).

**Lemma 4.5.** Let \( A \) be a C-algebra. Then for \( a, b \in A \):

(i) \( a \lor b = b \lor a \) if and only if \( S_{a \lor b} = S_a \cap S_b \)

(ii) \( S_{a \lor b} = \text{Sup}\{S_a, S_b\} \) in the poset \( \{S_x \mid x \in A\}, \subseteq \), then \( a \land b = b \land a \).

The converse is not true.

**Proof.** (i) Suppose that \( a \lor b = b \lor a \). Then clearly \( S_{a \lor b} \subseteq S_a \cap S_b \). By Theorem 4.2(ii) \( S_a \cap S_b \subseteq S_{a \lor b} \). Hence \( S_{a \lor b} = S_a \cap S_b \). Conversely assume that \( S_{a \lor b} = S_a \cap S_b \). Clearly \( a \lor b \in S_{a \lor b} = S_a \cap S_b \). Therefore \( a \lor b \in S_b \Rightarrow a \lor b = b \lor t \) for some \( t \in A \). Now \( b \lor a = b \lor a \lor b = b \lor b \lor t = b \lor t = a \lor b \) (ii) Assume that \( a, b \in A \) and \( S_{a \lor b} = \text{Sup}\{S_a, S_b\} \). Then \( S_{a \lor b} = S_{b \lor a} \) and hence \( a \land b \in S_{a \lor b} = S_{b \lor a} \). Therefore \( a \land b = (b \land a) \lor t \) for some \( t \in A \).
Now \((b \land a) \lor (a \land b) = (b \land a) \lor ((b \land a) \lor t) = (b \land a) \lor t = a \land b\). Similarly we can prove that \((a \land b) \lor (b \land a) = b \land a\). Hence \(a \land b = b \land a\). The converse need not be true, for example for the C-algebra \(C\), \(S_u = \{U\}, S_T = \{T\}\) and \(U \lor T = T \land U\). But \(S_{U \lor T} (= S_u)\) is not an upper bound of \(\{S_u, S_T\}\).

Now we prove \(\mathfrak{B}_{S(A)} = \{S_a \mid a \in B(A)\}\) is a Boolean algebra under set inclusion.

**Theorem 4.6.** Let \(<A, \land, \lor, \leq'>\) be a C-algebra with \(T\). Then \(\mathfrak{B}_{S(A)} = \{S_a \mid a \in B(A)\}\) is a Boolean algebra under set inclusion.

**Proof.** Clearly \((\mathfrak{B}_{S(A)}, \subseteq)\) is a partially ordered set under inclusion. First we show for \(a, b \in B(A)\), \(S_{a \lor b}\) is the infimum of \(\{S_a, S_b\}\) and \(S_{a \land b}\) is the supremum of \(\{S_a, S_b\}\) for all \(a, b \in B(A)\). Let \(a, b \in B(A)\). Then \(a \land b = b \land a\) and \(a \lor b = b \lor a\). Hence by the above Lemma 4.5, \(S_{a \lor b}\) is the infimum of \(\{S_a, S_b\}\). Let \(t \in S_a\). Then \(t = a \lor x\) for some \(x \in A\). Now \(t = a \lor x = (a \land (a \lor b)) \lor x = (a \land b) \lor x = (a \lor b) \lor x \in S_{a \lor b}\). Similarly \(S_b \subseteq S_{b \land a} = S_{a \lor b}\). Therefore \(S_{a \lor b}\) is an upper bound of \(S_a, S_b\). Suppose \(c\) is an upper bound of \(S_a, S_b\). Suppose \(c\) is an upper bound of \(S_a, S_b, t \in S_{a \lor b}\). Then \(t = (a \land b) \lor x\) for some \(x \in A\). Now \(t = (a \land b) \lor x = (a \lor x) \land (a' \lor b \lor x) = (a \lor x) \land (b \lor a' \lor x) \in S_c\) (since \(a \lor x \in S_a \subseteq S_c\), \(b \lor a' \lor x \in S_b \subseteq S_c\) and \(S_c\) is closed under \(\land\)). Therefore \(S_{a \lor b}\) is the supremum of \(\{S_a, S_b\}\). Denote the supremum of \(\{S_a, S_b\}\) by \(S_a \lor S_b\) and the infimum of \(\{S_a, S_b\}\) by \(S_a \land S_b\). Now \(S_T \land S_a = S_{T \land a} = S_T\) and \(S_F \lor S_a = S_{F \lor a} = S_F\). Therefore \(S_T\) is the least element and \(S_F\) is the greatest element of \((\mathfrak{B}_{S(A)}, \subseteq)\). Now for any \(a, b, c \in B(A), (S_a \lor S_b) \land S_c = S_{(a \lor b) \land c} = S_{(a \lor c) \land (b \lor c)} = S_{(a \lor c) \lor (b \lor c)} = (S_a \land S_c) \lor (S_b \land S_c)\). Also, \(S_a \land S_c = S_{a \land c} = S_T\) and \(S_a \lor S_c = S_{a \lor c} = S_F\). Therefore \((\mathfrak{B}_{S(A)}, \subseteq)\) is a complimented distributive lattice and hence it is a Boolean algebra.
Theorem 4.7. Let $A$ be a C-algebra with $T$. Define $\varphi : B(A) \rightarrow \mathcal{B}_{S(A)}$ by $\varphi(a) = s_a^x$ for all $a \in B(A)$. Then $\varphi$ is an isomorphism.

**Proof.** Let $a, b \in B(A)$. Then $\varphi(a \wedge b) = S_{(a, b)^x} = S_a^x \wedge S_b^x = \varphi(a) \wedge \varphi(b)$. $\varphi(a \vee b) = S_{(a, b)^x} = S_a^x \vee S_b^x = \varphi(a) \vee \varphi(b)$. $\varphi(a') = S_a^x' = (\varphi(a))'$. Clearly $\varphi$ is both one-one and onto. Hence $B(A) \cong \mathcal{B}_{S(A)}$.

In [3] we defined a partial ordering on a C-algebra by $x \leq y$ if and only if $y \wedge x = x$ and we studied the properties of this partial ordering. We gave a number of equivalent conditions in terms of this partial ordering for a C-algebra to become a Boolean algebra. In [4] we proved that, for each $x \in A$, $A_x = \{s \in A \mid s \leq x\}$ is itself a C-algebra under induced operations $\wedge, \vee$ and the unary operation is defined by $s^x = x \wedge s'$ we also observed that $A_x$ need not be an algebra of $A$ because the unary operation in $A_x$ is not the restriction of the unary operation. For each $x \in A$, we proved that $A_x$ is isomorphic to the quotient algebra $A_\theta_x$ where $\theta_x = \{(p, q) \in A \times A \mid x \wedge p = x \wedge q\}$. We can easily see that the C-algebras $S_x, A_x$ are different in general where $x \in A$.

Now, we prove that the set of all $A_x$'s where $a \in B(A)$ is a Boolean algebra under set inclusion. The following theorem can be proved analogous to Theorem 4.6.

**Theorem 4.8.** Let $A$ be a C-algebra with $T$. Then $\mathcal{B}_{R(A)} := \{A_x \mid a \in B(A)\}$ is a Boolean Algebra under set inclusion in which the supremum of $\{A_x, A_y\} = A_{x\wedge y}$ and the infimum of $\{A_x, A_y\} = A_{x\vee y}$.

The proof of the following theorem is analogous to that of Theorem 4.7.

**Theorem 4.9.** Let $A$ be a C-algebra with $T$. Define $f : B(A) \rightarrow \mathcal{B}_{R(A)}$ by $f(a) = A_x$ for all $a \in B(A)$. Then $f$ is an isomorphism.

The following corollary can be proved directly from Theorems 4.7 and 4.9.
Corollary 4.10. Let $A$ be a $C$-algebra with $T$. Then $B_{R(A)}, B(A)$ and $B_{S(A)}$ are isomorphic to each other.

References