Quivers of Bound Path Algebras and Bound Path Coalgebras

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Abstract. Algebras and coalgebras can be represented as quiver (directed graph), and from quiver we can construct algebras and coalgebras called path algebras and path coalgebras. In this paper we show that the quiver of a bound path coalgebra (resp. algebra) is the dual quiver of its bound path algebra (resp. coalgebra).

Keywords: *quiver; bound path algebras; bound path coalgebras.*

1 Introduction

Algebras can be represented as quiver (directed graph) and modules can be represented as quiver representation [1]. In [2] it is explained how to construct algebras from quivers called path algebras. In coalgebra representation theory, Chin [3] explained how to represent coalgebras as quiver, and also how to construct coalgebras form quivers called path coalgebras.

A finite dimensional algebra A over an algebraically closed field K is called *basic* if the quotient algebra of A modulo the Jacobson radical is isomorphic to a product of K as K-algebras. A theorem due to Gabriel says that a basic K-algebra is isomorphic to the factor algebra of the path algebra KQ_A by an admissible ideal, where Q_A is the quiver of A (see [1], [4]). Since any finite dimensional algebra is Morita equivalent to a uniquely determined basic algebra, it follows that any finite dimensional algebra A over an algebraically closed field is Morita equivalent to KQ_A modulo an admissible ideal.

Dually, a finite dimensional coalgebra C is called *pointed* if each simple sub¬coalgebra is of dimension one. As a dual of a result due to Gabriel, Chin and Montgomery [5] proved that any pointed coalgebra is isomorphic to a large subcoal¬gebra of the path coalgebra of the quiver C. Since any coalgebra is Morita-Takeuchi equivalent to a uniquely determined pointed coalgebra, it follows that any coalgebra C over an algebraically closed field is Morita-

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Takeuchi equivalent to a large subcoalgebra of the path coalgebra of quiver of C.

In [6] it is shown that the quiver of a path algebra (resp. path coalgebra) is the dual quiver of its path coalgebra (resp. algebra). In this paper we show that the same results hold for bound path algebras and bound path coalgebras. We show that the quiver of a bound path algebra (resp. bound path coalgebra) is the dual quiver of its bound path coalgebra (resp. bound path algebra), hence the dual of basic algebras are pointed coalgebras, and vice versa.

The paper is organized as follows: in section 2 we will give definition of path algebras and path coalgebras. In section 3 we will explain about dual path algebras and path coalgebras. Finally in the last section we will explain the main theorem about obtaining quivers of bound path algebras from quivers of bound path coalgebras and vice versa.

2 Path Algebras and Path Coalgebras

A quiver Q is a quadruple (Q_0, Q_1, s, t) where Q_0 is the set of vertices (points), Q_1 is the set of arrows and for each arrow $\alpha \in Q_1$, the vertices $s(\alpha)$ and $t(\alpha)$ are the source and the target of α, respectively, see [1]. If i and j are vertices, an (oriented) path in Q of length m from i to j is a formal composition of arrows.

$$p= \alpha_1\alpha_2 \ldots \alpha_m$$

where $s(\alpha_1) = i, t(\alpha_m) = j$ and $t(\alpha_{k-1}) = s(\alpha_k)$, for $k = 2, \ldots, m$. To any vertex $i \in Q_0$ we attach a trivial path of length 0, say e_i, starting and ending at i such that for any arrow α (resp. β) such that $s(\alpha) = i$ (resp. $t(\beta) = i$) then $e_i\alpha = \alpha$ (resp. $\beta e_i = \beta$). We identify the set of vertices and the set of trivial paths.

Let KQ be the K-vector space generated by the set of all paths in Q. Then KQ can be endowed with a structure of K-algebra with multiplication induced by concatenation of paths, that is,

$$(\beta_1\beta_2\ldots\beta_m)(\alpha_1\alpha_2\ldots\alpha_n) = \begin{cases} \beta_1\beta_2\ldots\beta_m\alpha_1\alpha_2\ldots\alpha_n & \text{if } t(\beta_m) = s(\alpha_1) \\ 0, & \text{otherwise} \end{cases}$$

KQ is called the path algebra of the quiver Q. The algebra KQ can be graded by

$$KQ = KQ_0 \oplus KQ_1 \oplus .. \oplus KQ_m \oplus ...,$$

where Q_m is the set of all paths of length m.

Definition 1 Let Q be a finite connected quiver. The ideal of path algebra KQ generated by arrows of Q is called *arrow ideal* and denoted by R_Q.

Definition 2 Let Q be a finite quiver and R_Q be the arrow ideal in path algebra KQ. An ideal I in KQ is *admissible* if there exists $m \geq 2$ such that

$$R^m Q \subseteq I \subseteq R^2 Q.$$

If I is an admissible ideal in KQ, (Q, I) is called bound quiver. The quotient algebra KQ/I is called *bound path algebra*.

The path algebra KQ can be viewed as a graded K-coalgebra with comultiplication induced by the decomposition of paths, that is, if $p = \alpha_1\alpha_2 ... \alpha_m$ is a path from the vertex i to the vertex j, then

$$\Delta(p) = e_i \otimes p + p \otimes e_j + \sum_{i=1}^{m-1} \alpha_1 ... \alpha_i \otimes \alpha_{i+1} ... \alpha_m = \sum_{\tau\upsilon = p} \tau \otimes \upsilon$$

and for a trivial path e_i we have $\Delta(ei) = ei \otimes ei$. The counit of KQ is defined by the formula

$$\varepsilon(\alpha) = \begin{cases} 1, \text{ if } \alpha \in Q_0 \\ 0, \text{ if } \alpha \text{ is a path of length } \geq 1 \end{cases}$$

The coalgebra $(KQ, \Delta, \varepsilon)$ is the path coalgebra of the quiver Q. For the convenience we denote by KQ the path algebra of Q and by CQ the path coalgebra of Q.

Definition 3 A subcoalgebra of a path coalgebra is said to be *admissible* if it contains the subcoalgebra generated by all vertices and all arrows, that is, $CQ_0 \oplus CQ_1$ (see [7]). A subcoalgebra C of a path coalgebra CQ is called a *relation subcoalgebra* (see [8]) if C is admissible and $C = \oplus_{a,b \in Q0} C \cap CQ(a,b)$ where $CQ(a,b)$ is the subspace generated by all paths starting at a and ending at b.

3 Dual Path Algebras and Path Coalgebras

3.1 Dual Path Coalgebras

Let Q be a quiver and CQ and KQ be the corresponding path coalgebra and path algebra, respectively. A basis of K-vector space $D(CQ)$ $(D(KQ))$ consists of p^* for all path p, $q \in Q$, where $p^*(q) = \delta_{pq}$, with δ is the delta Kronecker. Denote by $e_1, \cdots, e_n$ the trivial paths in Q. We define the multiplication map $\mu : D(CQ) \otimes D(CQ) \rightarrow D(CQ)$ and the unit map $\eta : K \rightarrow D(CQ)$ as following, for all p^*, q^* basis $D(CQ)$,

$$\mu(p^* \otimes q^*) = \begin{cases} (qp)^* \text{ if } t(q) = s(p) \\ \quad 0 \text{ otherwise} \end{cases}$$

$$\eta(1) = e_1^* + e_2^* + \cdots + e_n^*.$$

Lemma 4 *The multiplication μ and the unit η satisfy*

1. $\mu(id \otimes \mu) = \mu(\mu \otimes id)$
2. $(id \otimes \mu)(\eta \otimes id) = (\eta \otimes id)(id \otimes \mu)$

Proof

1. It is enough to show for basis elements of $D(CQ)$. Let p^*, q^* and r^* in basis of $D(CQ)$ where rqp is a path. Then

   $\mu(id \otimes \mu)(p^* \otimes q^* \otimes r^*) = \mu(p^* \otimes (rq)^*) = (rqp)^*$
   $\mu(\mu \otimes id)(p^* \otimes q^* \otimes r^*) = \mu((qp)^* \otimes r^*) = (rqp)^*$

2. Let q^* and r^* in basis of $D(CQ)$ where rq is a path.

$$((id \otimes \mu) \circ (\eta \otimes id))(1 \otimes q^* \otimes r^*) = (id \otimes \mu)\left(\left(\sum_{i-1}^{n} e_i^*\right) \otimes q^* \otimes r^*\right) = \left(\sum_{i=1}^{n} e_i^*\right) \otimes (rq)^* = (rq)^*$$

$$((\eta \otimes id) \circ (id \otimes \mu))(1 \otimes q^* \otimes r^*) = (\eta \otimes id)(1 \otimes (rq)^*) = \left(\sum_{i=1}^{n} e_i^*\right) \otimes (rq)^* = (rq)^* \text{ QED}$$

Corollary 5 $D(CQ)$ is a path algebra.

3.2 Dual Path Algebras

We define the comultiplication map $\Delta : D(KQ) \otimes D(KQ) \to D(KQ)$ as following

$$\Delta(p*) = \sum_{qr=p} r* \otimes q*$$

and for a trivial path e_i* we have $\Delta(e_i*) = e_i* \otimes e_i*$. The counit map ε : $D(KQ) \to K$ is defined by the formula

$$\varepsilon(\alpha*) = \begin{cases} 1, \text{ if } \alpha \in Q_0 \\ 0, \text{ if } \alpha \text{ is a path of length} \geq 1 \end{cases}$$

Lemma 6 The comultiplication Δ and the counit ε satisfy

1. $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$
2. $(id \otimes \varepsilon)\Delta = (\varepsilon \otimes id)\Delta$.

Proof Let $p^* \in D(KQ)$.

$$(id \otimes \Delta)\Delta(p*) = (id \otimes \Delta)\left(\sum_{qr=p} r* \otimes p*\right) = \sum_{qr=p} r* \otimes \left(\sum_{uv=q} v* \otimes u*\right) = \sum_{uvr=p} r* \otimes v* \otimes u*$$

$$(\Delta \otimes id)\Delta(p*) = (\Delta \otimes id)\left(\sum_{qr=p} r* \otimes p*\right) = \sum_{uq=p}\left(\sum_{vr=q} r* \otimes v*\right) \otimes u* = \sum_{uvr=p} r* \otimes v* \otimes u*$$

Moreover

$$(id \otimes \varepsilon)\Delta(p*) = (id \otimes \varepsilon)\left(\sum_{qr=p} r* \otimes q*\right) = p*$$

$$(\varepsilon \otimes id)\Delta(p*) = (\varepsilon \otimes id)\left(\sum_{qr=p} r* \otimes q*\right) = p*$$

QED

Corollary 7 *D(KQ)* is a path coalgebra.

3.3 Opposite Quivers

For a quiver Q with set of vertices $\{1, 2, \ldots, n\}$ we denote by Q^{op} the quiver having the same set of vertices. For each arrow $\alpha : i \to j$ in Q there is an arrow $\alpha^{op}: j \to i$ in Q^{op}. For a path $p = \alpha_1\alpha_2 \ldots \alpha_m$ define the path p^{op} as $p = \alpha_m^{op}\, \alpha_{m-1}^{op} \ldots \alpha_1^{op}$.

We define a bilinear map $(-, -) : CQ \times KQ^{op} \to K$ by $(v,w) = \delta_{vw}$ (the Kronecker's delta) for any two paths $v,w \in Q$. This bilinear map defined above is degenerate in the following sense:

1. if $(v, w) = 0$ for all $v \in CQ$, then $w = 0$.
2. if $(v, w) = 0$ for all $w \in KQ^{op}$, then $v = 0$.

This means that there exist two injective linear maps $\sigma: CQ \to D(KQ^{op})$ and $\tau: KQ^{op} \to D(CQ)$ defined by $\sigma(v)(w) = (v,w)$ and $\tau(w)(v) = (v,w)$, for all $v \in CQ$ and $w \in KQ^{op}$.

Lemma 8 If Q is any quiver, then

1. the injective morphism $KQ^{op} \to D(CQ)$ is a morphism of algebras,
2. the injective morphism $CQ \to D(KQ^{op})$ is a morphism of coalgebras.

Proof These are consequences of Corollary 5 and Corollary 7 QED

4 Quivers with Relations

Let $v \in CQ$, the orthogonal subspace to v is the set $v^{\perp} = \{f \in D(CQ) : f(v) = 0\}$. More generally, for any subset $S \subseteq Q$, we define the orthogonal subspace to S to be the space

$$S^{\perp} = \{f \in D(CQ) : f(S) = 0\}.$$

Since KQ^{op} can be embedded in $D(CQ)$ then we may consider the orthogonal subspace to S in KQ^{op}

$$S^{\perp KQ^{op}} = S^{\perp} \cap KQ^{op} = \{w \in KQ^{op} : (S, w) = 0\}.$$

Reciprocally, for any subset $T \subseteq D(CQ)$ the orthogonal subspace to T in CQ, is defined by the formula

$$T^{\perp CQ} = \{v \in CQ : f(v) = 0 \text{ for all } f \in T\}.$$

And if $T \subseteq KQ^{op}$, then we write

$$T^{\perp CQ} = \{v \in CQ : (v, w) = 0 \text{ for all } w \in T\}.$$

For simplicity we write $\perp$ instead of $\perp CQ$.

Definition 9 Let (Q^{op}, I) be a quiver with relations. The *bound path coalgebra* of (Q^{op}, I) is defined by the subspace of CQ,

$$C(Q, I) = \{a \in CQ : (a, I) = 0\}.$$

Lemma 10 Let Q be a quiver and C a relation subcoalgebra of CQ. Then $C^{\perp KQ^{op}}$ is an admissible ideal of KQ.

Proof See [9, Corollary 4.3].

Proposition 11 Let Q^{op} be a quiver and I an admissible ideal of KQ^{op}, then $C(Q, I) = I^{\perp}$ is a relation subcoalgebra of CQ.

Proof See [9].

Lemma 12 Let Q be a finite quiver and C a finite dimensional relation subcoalgebra of CQ, then there exists an admissible ideal I of KQ^{op} such that $C = C(Q, I)$.

Proof Choose $I = C^{\perp KQ^{op}}$. By [10, Theorem 2.2.1], $C = \left(C^{\perp KQ^{op}}\right)^{\perp} = I^{\perp}$. And the later is C(Q,I) by Proposition 11. QED

Now we will state the main theorem which is is a generalization of the results in [6].

Theorem 13 (Main Theorem) Let Q be a finite quiver.

1. If I is an admissible ideal of KQ^{op} then $D(KQ^{op}/I) \cong I^{\perp} = C(Q, I)$.
2. If C is a finite dimensional relation subcoalgebra of CQ then $D(C) \cong KQ^{op} / C^{\perp KQ^{op}}$.

Proof

1. Let $f \in D(KQ^{op}/I)$. Then it is clear $f \in D(KQ^{op})$such that $(f, I) = 0$, hence $f \in I^{\perp}$. Conversely, if $f \in I^{\perp}$, then we can identify f with $f' \in D(KQ^{op}/I)$, where $f'(\underline{p}) = f(p)$ for all $\underline{p} \in KQ^{op}/I$. This completes the proof.

2. By Lemma 12 and the statement above,

$$C = C(Q,I) = I^{\perp} \cong D(KQ^{op}/I) \text{ where } I = C^{\perp KQ^{op}}.$$

Hence $D(C) \cong KQ^{op}/I$.

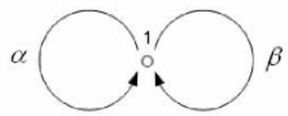
QED

Corollary 14 Let C be a relation subcoalgebra of CQ and $D(C)$ be its dual. Then $D(C)$ is isomorphic to the bound path algebra of quiver (Q^{op}, I),i.e., the quiver of dual path subcoalgebra $D(C)$ can be obtained from the quiver Q by reversing arrows.

Corollary 15 Let KQ/I be a bound path algebra where I is an admissible ideal of KQ, and $D(KQ/I)$ be its dual. Then $D(KQ/I)$ is isomorphic to the bound path coalgebra of quiver (Q^{op}, I) i.e., the quiver of dual bound path algebra $D(KQ/I)$ can be obtained from the quiver Q by reversing arrows.

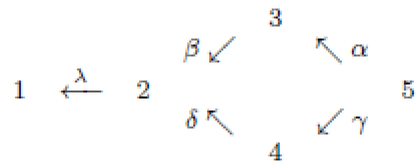
We give here some examples:

Example 16 Let Q be the quiver

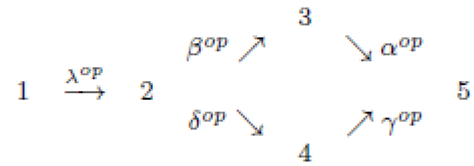


The ideal I generated by $\alpha\beta - \beta\alpha, \beta^2, \alpha^2$ is an admissible ideal. The bound path algebra KQ/I is 4-dimensional with basis $\{\overline{e_1}, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta}\}$. The dual of KQ/I is $C(Q^{op}, I)$, the relation subcoalgebra of CQ^{op} where Q^{op} is the same quiver as Q.

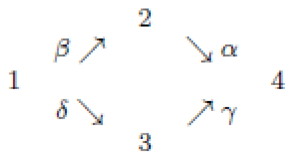
Example 17 Let Q be the quiver



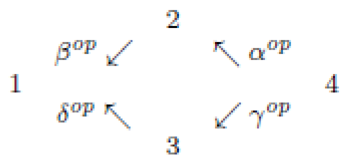
bound by the relation $\alpha\beta = \gamma\delta$. Then the dual of KQ/I is the relation subcoalgebra $C(Q^{op}, I)$ where Q^{op} is the quiver



Example 18 Let Q be the quiver



and C be the relation subcoalgebra generated by $\{\beta\alpha + \gamma\delta\}$. Then $D(C)$, the dual of C is $KQ^{op} / C^{\perp KQ^{op}}$ where Q^{op} is the following quiver:



As conclusion, from the relation between path coalgebras and path algebras, we can study more on coalgebras and algebras using their quivers. We may also conclude that dual of basic algebras are pointed coalgebras and dual of pointed coalgebras are basic algebras.

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