# Locating-Chromatic Number of Amalgamation of Stars

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**Abstract.** Let $G$ be a connected graph and $c$ a proper coloring of $G$. For $i = 1, 2, \ldots, k$ define the color class $C_i$ as the set of vertices receiving color $i$. The color code $c_\Pi(v)$ of a vertex $v$ in $G$ is the ordered $k$-tuple $(d(v,C_1),\ldots,d(v,C_k))$ where $d(v,C_i)$ is the distance of $v$ to $C_i$. If all distinct vertices of $G$ have distinct color codes, then $c$ is called a locating-coloring of $G$. The locating-chromatic number of graph $G$, denoted by $\chi_L(G)$ is the smallest $k$ such that $G$ has a locating coloring with $k$ colors. In this paper we discuss the locating-chromatic number of amalgamation of stars $S_{k,m}$. $S_{k,m}$ is obtained from $k$ copies of star $K_{1,m}$ by identifying a leaf from each star. We also determine a sufficient condition for a connected subgraph $H \subseteq S_{k,m}$ satisfying $\chi_L(H) \le \chi_L(S_{k,m})$.

**Keywords:** *amalgamation of stars; color code; locating-chromatic number.*

## 1 Introduction

Let $G$ be a finite, simple, and connected graph. Let $c$ be a proper coloring of a connected graph $G$ using the colors $1, 2, \ldots, k$ for some positive integer $k$, where $c(u) \neq c(v)$ for adjacent vertices $u$ and $v$ in $G$. Thus, the coloring $c$ can be considered as a partition $\prod$ of $V(G)$ into color classes (independent sets) $C_1, C_2, \ldots, C_k$, where the vertices of $C_i$ are colored by $i$ for $1 \le i \le k$. The *color code* $c_\Pi(v)$ of a vertex $v$ in $G$ is the ordered $k$-tuple $(d(v,C_1),\ldots,d(v,C_k))$ where $d(v,C_i) = \min\{d(v,x) \mid x \in C_i\}$ for $1 \le i \le k$. If all distinct vertices of $G$ have distinct color codes, then $c$ is called a *locating-coloring* of $G$. A *minimum locating-coloring* uses a minimum number of colors and this number is called the *locating-chromatic number* of graph $G$, denoted by $\chi_L(G)$.

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The following definition of an amalgamation of graphs is taken from [3]. For $i=1,2,\ldots,k$, let $G_i$ be a graph with a fixed vertex $v_{oi}$. The *amalgamation* $\mathrm{Amal}(G_i, v_{oi})$ is a graph formed by taking all the $G_i$'s and identifying their fixed vertices. In this paper, we consider the amalgamation of stars. More precisely, for $i=1,2,...,k$, let $G_i = K_{1,n_i}, n_i \geq 1$ where $v_{oi}$ be any leaf (a vertex of degree 1) of $K_{1,n_i}$. We denote the amalgamation of $k$ stars $K_{1,n_i}$ by $S_{k,(n_1,n_2,...,n_k)}, k \geq 2$. We call the identified vertex as the *center* (denoted by $x$), the vertices of distance 1 from the center as the *intermediate vertices* (denoted by $l_i$ ; $i=1,2,...,k$), and the $j$-th leaf of the intermediate vertex $l_i$ by $l_{ij}$ $(j=1,2,...,m-1)$. In particular, when $n_i = m$, $m \geq 1$ for all $i$, we denote the amalgamation of $k$ isomorphic stars $K_{1,m}$ by $S_{k,m}$.

The locating-chromatic number was firstly studied by Chartrand *et al*. [1]. They determined the locating-chromatic numbers of some well-known classes such as paths, cycles, complete multipartit graphs and double stars. Apart of paths and double stars, the locating-chromatic number of other trees are also considered by Chartrand *et al*. [2]. They constructed a tree of order $n \geq 5$ with the locating- chromatic number $k$, where $k \in \{3,4,...,n-2,n\}$. They also showed that no tree on $n$ vertices with locating-chromatic number $n-1$.

Based on the previous results, locating-chromatic number of amalgamation of stars have not been studied. Motivated by this, in this paper we determine the locating-chromatic number of amalgamation of stars.

Beside that, we also discuss the monotonicity property of the locating-chromatic number for the class of amalgamation of stars. Clearly, the locating-chromatic number of a star $K_{1,n}$ is $n+1$, for any $n$ (since all vertices must have different color codes). Since any connected subgraph $H$ of $K_{1,n}$ is also a star with small size, then we clearly have $\chi_L(H) \leq \chi_L(K_{1,n})$. However in general for any connected subgraph $H \subseteq G$, the locating-chromatic number of $H$ may not be necessarily smaller or equal to the locating-chromatic number of $G$.

In this paper, we also investigate the monotonicity property of the locating-chromatic number for amalgamation of stars, $S_{k,m}$. We derive a sufficient condition for a connected subgraph $H \subseteq S_{k,m}$ satisfying $\chi_L(H) \leq \chi_L(S_{k,m})$.

The following results were proved by Chartrand et al. in [1] . The set of neighbours of a vertex $v$ in $G$ is denoted by $N(v)$.

**Theorem 1.1.** *Let $c$ be a locating-coloring in a connected graph $G$. If $u$ and $v$ are distinct vertices of $G$ such that $d(u,w)=d(v,w)$ for all $w\in V(G)-\{u,v\}$, then $c(u)\neq c(v)$. In particular, if $u$ and $v$ are non adjacent vertices of $G$ such that $N(u)=N(v)$, then $c(u)\neq c(v)$.*

**Corollary 1.1.** *If $G$ is a connected graph containing a vertex adjacent to $k$ leaves of $G$, then $\chi_L(G)\geq k+1$.*

## 2 Main Results

We first prove some lemmas regarding the properties of locating-chromatic number of amalgamation of stars. From now on $S_{k,m}$ denotes the amalgamation of $k$ isomorphic stars $K_{1,m}$.

**Lemma 2.1.** *For $k\geq 2, m\geq 2$, let $c$ be a proper coloring of $S_{k,m}$, using at least $m$ colors. The coloring $c$ is a locating-coloring if and only if $c(l_i)=c(l_n), i\neq n$ implies $\{c(l_{ij})\mid j=1,2,...,m-1\}$ and $\{c(l_{nj})\mid j=1,2,...,m-1\}$ are distinct.*

*Proof.* Let $P=\{c(l_{ij})\mid j=1,2,...,m-1\}$ and $Q=\{c(l_{nj})\mid j=1,2,...,m-1\}$. Let $c$ be a locating-coloring of $S_{k,m}$, $k\geq 2, m\geq 2$ using at least $m$ colors and $c(l_i)=c(l_n)$, for some $i\neq n$. Suppose that $P=Q$. Because $d(l_i,u)=d(l_n,u)$ for every $u\in V\setminus\left\{\{l_{ij}\mid j=1,2,...,m-1\}\cup\{l_{nj}\mid j=1,2,...,m-1\}\right\}$ then the color codes of $l_i$ and $l_n$ will be the same. So $c$ is not a locating-coloring, a contradiction. Therefore $P\neq Q$.

Let $\prod$ be a partition of $V(G)$ into color classes with $|\prod|\geq m$. Consider $c(l_i)=c(l_n), i\neq n$. Since $P\neq Q$, there are color $x$ and color $y$ such that $(x\in P, x\notin Q)$ and $(y\in P, y\notin Q)$. We will show that color codes for every $v\in V(S_{k,m})$ is unique.

- Clearly, $c_{\prod}(l_i)\neq c_{\prod}(l_n)$ because their color codes differ in the $x$th-ordinate and $y$th-ordinate.

- If $c(l_{ij}) = c(l_{ns})$, for some $l_i \neq l_n$, we will show that $c_\Pi(l_{ij}) \neq c_\Pi(l_{ns})$. We divide into two cases.

  Case 1. If $c(l_i) = c(l_n)$ then by the premise of this theorem, $P \neq Q$. So $c_\Pi(l_{ij}) \neq c_\Pi(l_{ns})$.

  Case 2. Let $c(l_i) = r_1$ and $c(l_n) = r_2$, with $r_1 \neq r_2$. Then $c_\Pi(l_{ij}) \neq c_\Pi(l_{ns})$ because their color codes are different at least in the $r_1$th-ordinate and $r_2$th-ordinate.

- If $c(l_i) = c(l_{nj})$, $l_i \neq l_n$, then $c_\Pi(l_i)$ contains at least two components of value $1$, whereas $c_\Pi(l_{nj})$ contains exactly one component of value 1. Thus $c_\Pi(l_i) \neq c_\Pi(l_{nj})$.

- If $c(x) = c(l_{ij})$, then color code of $c_\Pi(x)$ contains at least two components of value $1$, whereas $c_\Pi(l_{ij})$ contains exactly one component of value 1. Thus $c_\Pi(x) \neq c_\Pi(l_{ij})$.

From all above cases, we see that the color code for each vertex in $S_{k,m}$ is unique, thus $c$ is a locating-coloring. $\square$

**Lemma 2.2.** *Let* $c$ *be a locating coloring of* $S_{k,m}$ *using* $m+a$ *colors and* $H(a) = (m+a-1)\binom{m+a-1}{m-1}$, $a \geq 0$. *Then* $k \leq H(a)$.

*Proof.* Let $c$ be a locating-coloring of $S_{k,m}$ using $m+a$ colors. For fixed $i$, let $c(l_i)$ be a color of intermediate vertex $l_i$ ,then the number color combinations can be used by $\{l_{ij} \mid j = 1, 2, ..., m-1\}$ is $\binom{m+a-1}{m-1}$. Because one color is used for coloring the center $x$, there are $(m+a-1)$ colors for $l_i$, for every $i = 1, 2, ..., k$. By Lemma 2.1, the maximum number of $k$ is $(m+a-1)\binom{m+a-1}{m-1} = H(a)$. So $k \leq H(a)$. $\square$

The main result of this paper concerns about locating-chromatic number of $S_{k,m}$.

**Theorem 2.1.** *For* $a \geq 0, k \geq 2, m \geq 2$, *let* $H(a) = (m+a-1)\binom{m+a-1}{m-1}$. *Then,*

$$\chi_L(S_{k,m}) = \begin{cases} m & \text{for} \quad 2 \leq k \leq H(0),\ m \geq 3, \\ m+a & \text{for} \quad H(a-1) < k \leq H(a), a \geq 1. \end{cases}$$

*Proof.* First, we determine the trivial lower bound. By Corollary 1.1, each vertex $l_i$ is adjacent to $(m-1)$ leaves, for $i = 1, 2, ..., k$. Thus, $\chi_L(S_{k,m}) \geq m$.

Next, we determine the upper bound of $\chi_L(S_{k,m})$ for $2 \leq k \leq H(0) = m-1$. Let $c$ be a coloring of $V(S_{k.m})$ using $m$ colors. Without loss of generality, we can assign $c(x) = 1$ and $c(l_i) = i+1$ for $i = 1, 2, ..., k$. To make sure that the leaves will have distinct color code, we assign $\{l_{ij} \mid j = 1, 2, ..., m-1\}$ by $\{1, 2, ..., m\} \setminus \{i+1\}$ for any $i$. Then, by Lemma 2.1, we have that $c$ is a locating-coloring. Thus $\chi_L(S_{k,m}) \leq m$.

Next, we shall improve the lower bound for the case of $k$ such that $H(a-1) < k \leq H(a), a \geq 1$. Since $k > H(a-1)$ then by Lemma 2.2, $\chi_L(S_{k,m}) \geq m+a$. On the other hand if $k > H(a)$ then by Lemma 2.2, $\chi_L(S_{k,m}) \geq m+a+1$. Thus $\chi_L(S_{k,m}) \geq m+a$ if $H(a-1) < k \leq H(a)$.

Next, we determine the upper bound of $\chi_L(S_{k,m})$ for $H(a-1) < k \leq H(a)$, $a \geq 1$. Without loss of generality, let $c(x) = 1$ and color the intermediate vertices $l_i$ by $2, 3, ..., m+a$ in such a way that the number of the intermediate vertices receiving the same color $t$ does not exceed $\binom{m+a-1}{m-1}$, for any $t$. We are able to do so because $H(a-1) < k \leq H(a)$. Therefore, if $c(l_i) = c(l_n)$, $i \neq n$ then we can manage $\{c(l_{ij}) \mid j = 1, 2, ..., m-1\} \neq \{c(l_{nj}) \mid j = 1, 2, ..., m-1\}$. By Lemma 2.1, $c$ is a locating-coloring on $S_{k,m}$. So $\chi_L(S_{k,m}) \leq m+a$ for $H(a-1) < k \leq H(a)$. □

The following figures show minimum locating-colorings on $S_{4,6}$ and $S_{9,3}$ .

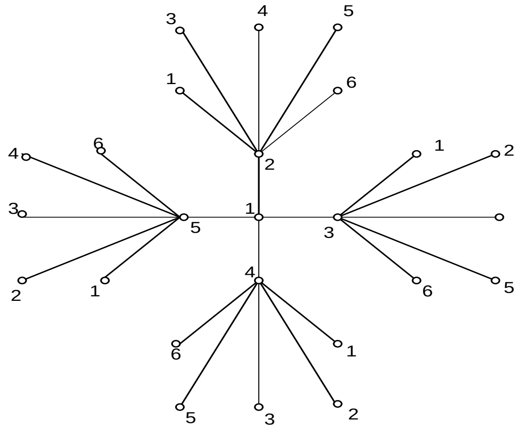

**Figure 1** A minimum locating-coloring of $S_{4,6}$ .

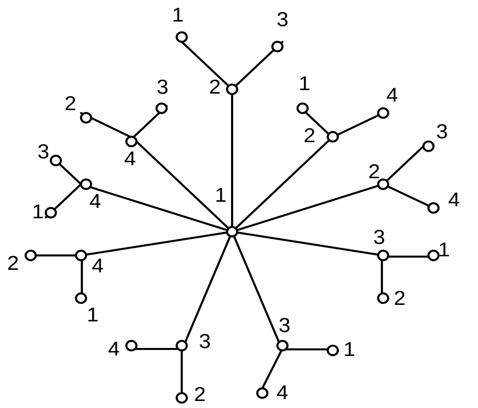

**Figure 2** A minimum locating-coloring of $S_{9,3}$ .

Next, we discuss the monotonicity property of locating-chromatic number for the amalgamation of stars.

**Theorem 2.2** *If* $2 \le k \le m-1$, *then* $\chi_L(G) \le \chi_L(S_{k,m})$ *for every* $G \subseteq S_{k,m}$ *and* $G \ne K_{1,m}$ .

*Proof.* Let $c$ be a minimum locating-coloring of $S_{k,m}$ obtained from Theorem 2.1. Let $G$ be any connected subgraph of $S_{k,m}$ . Define a coloring $c'$ on $G$ by preserving colors used in $S_{k,m}$ for the corresponding vertices, namely $c'(v') = c(v)$ if $v$ is the corresponding vertex of $v'$ in $S_{k,m}$. We show that $c'$ is a locating-coloring of $G$.

If there exist $l_i$, $l_n$ such that $\{c'(l_{ij}) \mid j = 1,2,...,r\} = \{c'(l_{nj}) \mid j = 1,2,...,r\}$, with $1 \le r \le m-1$, then color codes of $l_{ij}$ and $l_{nj}$ for every $j = 1,2,3,...,m$ is unique because $c'(l_i) \ne c'(l_n)$ for every $l_i \ne l_n$. If $c'(l_i) = c'(l_{nj}) \ne c'(x)$, then the first component of $c'_\pi(l_i)$ has value 1, whereas for $c'_\pi(l_{nj})$ it has value 2. So color code of $l_i$ and $l_{nj}$ are different. Next, if $c'(x) = c'(l_{nj})$, $G \ne P_3$ then their color codes are different because $c'(l_i) \ne c'(l_n)$ for every $l_i \ne l_n$. For the case $G = P_3$, $v_i \in V(P_3)$ for each $i$ is colored by 1, 2, and 3 respectively. Because the color codes for every $v \in V(G)$ is unique, then $c'$ is a locating-coloring of $G$. So $\chi_L(G) \le \chi_L(S_{k,m})$ for every $G \subseteq S_{k,m}$, $G \ne K_{1,m}$. □

Let $S_{k,(n_1,n_2,...,n_k)} \subseteq S_{k,m}$. Define $A = \{i \mid n_i = 1\}$. For $k \ge m$, we must restrict subgraphs of $S_{k,m}$ so that satisfy monotonicity property.

**Theorem 2.3** *If* $k \ge m$ *and* $|A| \le \chi_L(S_{k,m}) - 1$ *then* $\chi_L(S_{k,(n_1,n_2,...,n_k)}) \le \chi_L(S_{k,m})$.

*Proof.* Let $k \ge m$ and from Theorem 2.1, we have that $\chi_L(S_{k,m}) = m + a$ for $H(a-1) < K \le H(a), a \ge 1$. Let $G = S_{k,(n_1,n_2,...,n_k)}$ be any subgraph obtained from $S_{k,m}$ with $1 \le n_i \le m$. If $2 \le n_i \le m$ for each $i$, then color vertices of $G$ follow the proof of Theorem 2.1. Clearly, the coloring of $G$ is a locating-coloring. Otherwise, we have $n_i = 1$ for some $i$, and so $|A| \ge 1$. If $|A| \le \chi_L(S_{k,m}) - 1$, then the center $x$ is given color 1, $l_i \in A$ for each $i$ is colored by $2,3,...,\chi_L(S_{k,m})$, respectively and the colors for the other vertices follow the proof of Theorem 2.1. Observe that the color codes of $l_i$ for each $l_i \in A$ has value 1 in the 1th-ordinate, 0 in the $i$th-ordinate, and 2 otherwise, these color codes are unique. For the remaining of the vertices, the color codes are also unique as proven in Theorem 2.1. As the result, the coloring of $G$ is a locating-coloring. So $\chi_L(S_{k,(n_1,n_2,...,n_k)}) \le \chi_L(S_{k,m})$. □

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