# *P*-, *I*-, *g*-, and *D*-Angles in Normed Spaces

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**Abstract**. The notion of angles is known in a vector space equipped with an inner product, but not well established in a vector space equipped only with a norm. In this note, we shall develop some notions of angles between two vectors in a normed space and discuss their properties.

**Keywords:** *angles; inner product spaces; normed spaces; orthogonality.*

## 1 Introduction

In an inner product space ($X$, $<\bullet,\bullet>$), the angle $A(x,y)$ between two nonzero vectors $x$ and $y$ in $X$ is usually given by

$$A(x,y) = \arccos \frac{< x, y >}{\| x \| . \| y \|},$$

where $\|x\| := <x,x>^{1/2}$ denotes the induced norm in $X$. One may observe that the angle $A(\bullet,\bullet)$ in $X$ satisfies the following basic properties (see [1]):

1.1 *Parallelism*: $A(x,y) = 0$ if and only if $x$ and $y$ are of the same direction; $A(x,y) = \pi$ if and only if $x$ and $y$ are of opposite direction.
1.2 *Symmetry*: $A(x,y) = A(y,x)$ for every $x$, $y$ in $X$.
1.3 *Homogeneity*: $A(ax,by) = A(x,y)$ if $ab > 0$; $A(ax,by) = \pi - A(x,y)$ if $ab < 0$.
1.4 *Continuity*: If $x_n \to x$ dan $y_n \to y$ (in norm), then $A(x_n,y_n) \to A(x,y)$.

Now suppose ($X$, $\|\bullet\|$) is a (real) normed space. As it is known, not all normed spaces are inner product spaces. For instance, the space $l^p = l^p(\mathbf{R})$, $1 \le p < \infty$, consisting of all real sequences $x := (\xi_k)$ with $\Sigma\, |\xi_k|^p < \infty$, is a normed space with norm $\|x\|_p := \left[\sum |\xi_k|^p\right]^{1/p}$, but not an inner product space, except for $p = 2$.

In a normed space we often talk about the length of a vector or the distance between two vectors. Also, there are several ways that enable us to define orthogonality between two vectors. In this note, we shall develop some notions of angles between two vectors and discuss their properties. Preliminary results

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have been presented in a national seminar at Universitas Negeri Yogyakarta on November 18th, 2005 [2].

## 2 *P*- and *I*-angle

Although the notion of angles in a normed space may have been developed by some researchers, the literatures on this subject are very limited. As far as we know, the notions of angles discussed here have never been found before in the literatures, except for the *g*-angle [3].

Let $(X, \|\bullet\|)$ be a (real) normed space. Before we come to the notions of angles between two vectors in $X$, let us recall the following notions of *P*- and *I*-orthogonality (see [4] or [5]):

2.1 *P-orthogonality*: $x$ is *P*-orthogonal to $y$, denoted by $x \perp_P y$, if and only if

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 .$$

2.2 *I-orthogonality*: $x$ is *I*-orthogonal to $y$, denoted by $x \perp_I y$, if and only if

$$\|x+y\| = \|x-y\| .$$

Inspired by these two notions of orthogonality, we define the following notions of angles in $X$:

2.3 *P-angle* between two nonzero vectors $x$ and $y$, denoted by $A_P(x,y)$, is given by:

$$A_P(x,y) := \arccos \frac{\|x\|^2 + \|y\|^2 - \|x-y\|^2}{2\|x\|.\|y\|}.$$

2.4 *I-angle* between two nonzero vectors $x$ and $y$, denoted by $A_I(x,y)$, is given by:

$$A_I(x,y) := \arccos \frac{\|x+y\|^2 - \|x-y\|^2}{4\|x\|.\|y\|}.$$

Note that $A_P(x,y) = \frac{1}{2}\pi$ if and only if $x \perp_P y$, and $A_I(x,y) = \frac{1}{2}\pi$ if and only if $x \perp_I y$. In an inner product space$(X, \langle\bullet,\bullet\rangle)$, *P*- and *I*-angle coincide with the usual angle $A(x,y)$ because

$$\tfrac{1}{2}.[\,\|x\|^2 + \|y\|^2 - \|x-y\|^2\,] = \langle x,y \rangle$$

from the Cosine Law, and

$$\tfrac{1}{4}.[\| x + y \|^2 - \| x - y \|^2 ] = < x,y >$$

from the Polarization Identity.

The two facts below describe the properties of *P*- and *I*-angle in $(X, \|\bullet\|)$.

**Fact 2.5** *P*-angle satisfies the following properties:

(a) If $x$ and $y$ are of the same direction, then $A_P(x,y) = 0$; if $x$ and $y$ are of opposite direction, then $A_P(x,y) = \pi$ (part of parallelism property).
(b) $A_P(x,y) = A_P(y,x)$ for every $x$ and $y$ in $X$ (symmetry property).
(c) $A_P(ax,ay) = A_P(x,y)$ for every $x$, $y$ in $X$ and $a$ in $\mathbf{R}$ (part of homogeneity property).
(d) If $x_n \to x$ dan $y_n \to y$ (in norm), then $A_P(x_n,y_n) \to A_P(x,y)$ (continuity property).

*Proof.*
(a) Recall that $\|kx\| = |k|.\|x\|$ for every $x$ in $X$ and $k$ in $\mathbf{R}$. Now if $x$ and $y$ are of the same direction, then $y = kx$, with $k > 0$, and hence $A_P(x,y) = \arccos(1) = 0$. If $x$ and $y$ are of opposite direction, then $y = kx$, with $k < 0$, and hence $A_P(x,y) = \arccos(-1) = \pi$.
(b) Simmetry property is clearly satisfied since $\|x - y\| = \|y - x\|$ for every $x$ and $y$ in $X$.
(c) This part is clear because the factor $a^2$ is cancelled from both numerator and denominator when we compute the arccos.
(d) The continuity follows from the continuity of the norm and the arccos.

**Fact 2.6** *I*-angle satisfies the following properties:
(a) If $x$ and $y$ are of the same direction, then $A_I(x,y) = 0$; if $x$ and $y$ are of opposite direction, then $A_I(x,y) = \pi$ (part of parallelism property).
(b) $A_I(x,y) = A_I(y,x)$ for every $x$ and $y$ in $X$ (symmetry property).
(c) $A_I(ax,ay) = A_P(x,y)$ and $A_I(ax,-ay) = \pi - A_I(x,y)$ for every $x$, $y$ in $X$ and $a$ in $\mathbf{R}$ (part of homogeneity property).
(d) If $x_n \to x$ dan $y_n \to y$ (in norm), then $A_I(x_n,y_n) \to A_I(x,y)$ (continuity property).

*Proof.*
(a) If $x$ and $y$ are of the same direction, then $y = kx$, with $k > 0$, and hence $A_I(x,y) = \arccos(1) = 0$. If $x$ and $y$ are of opposite direction, then $y = kx$, with $k < 0$, and hence $A_I(x,y) = \arccos(-1) = \pi$.

(b) Simmetry property is obviously satisfied since $||x + y|| = ||y + x||$ and $||x - y|| = ||y - x||$ for every $x$ and $y$ in $X$.
(c) The first part is clear because the factor $a^2$ is cancelled from both numerator and denominator when we compute the arccos. For the second part, we use the fact that if $A_I(ax,ay) = \arccos u(x,y)$, then $A_I(ax,-ay) = \arccos(-u(x,y))$, from which we get $A_I(ax,-ay) = \pi - A_I(ax,ay) = \pi - A_I(x,y)$.
(d) As for *P*-angle, the continuity of *I*-angle follows from the continuity of the norm and the arccos.

*Remark*. The homogeneity property is not satisfied by *P*-angle and *I*-angle. For example, in the space $l^1$, one may take $x = (3,6,0,0,0,\dots)$ and $y = (8,-4,0,0,0,\dots)$, for which $A_P(x,y) = \frac{1}{2}\pi$ (that is, $x \perp_P y$) but $A_P(x,2y) \neq \frac{1}{2}\pi$. Similarly, if one takes $x = (2,1,0,0,0,\dots)$ and $y = (1,-2,0,0,0,\dots)$, then $A_I(x,y) = \frac{1}{2}\pi$ but $A_I(x,2y) \neq \frac{1}{2}\pi$.

## 3 *g*- and *D*-angle

We shall now discuss two different notions of angles, namely *g*- and *D*-angles. The former is due to Milicic and is related to *g*-orthogonality [3], while the latter is related to *D*-orthogonality [6].

Let $(X,||\bullet||)$ be a normed space. The functional $g : X^2 \to \mathbf{R}$ defined by

$$g(x,y) := \tfrac{1}{2}\,||x||\,[\tau_+(x,y) + \tau_-(x,y)]$$

where $\tau_\pm(x,y) := \lim_{t\to\pm 0} \dfrac{||x+ty|| - ||x||}{t}$, satisfies

(i) $g(x,x) = ||x||^2$ for every $x$ in $X$;
(ii) $g(ax,by) = ab.g(x,y)$ for every $x$, $y$ in $X$ and $a$, $b$ in $\mathbf{R}$;
(iii) $g(x,x+y) = ||x||^2 + g(x,y)$ for every $x$, $y$ in $X$;
(iv) $|g(x,y)| \leq ||x||.||y||$ for every $x$, $y$ in $X$.

If, in addition, the functional $g(x,y)$ is linear in $y$, then $g$ is called a *semi-inner product* on $X$. For example, the functional

$$g(x,y) := ||x||_p^{2-p} \sum |\xi_k|^{p-1}\,\mathrm{sgn}(\xi_k)\eta_k, \quad x = (\xi_k), y = (\eta_k) \in l^p,$$

is a semi-inner product on $l^p$, $1 \leq p < \infty$. Using a semi-inner product $g$, one can define *g*-orthogonality on $X$ as follows:

3.1 *g-orthogonality*: $x$ is $g$-orthogonal to $y$, denoted by $x \perp_g y$, if and only if $g(x,y) = 0$.

Note that in an inner product space $(X,\langle\bullet,\bullet\rangle)$, the functional $g(x,y)$ is identical with the inner product $\langle x,y\rangle$, and so $g$-orthogonality coincides with the usual orthogonality (see [7] for its verification).

Related to $g$-orthogonality, we can define $g$-angle in $X$ as follows.

3.2 *g-angle* between two vectors $x$ and $y$, denoted by $A_g(x,y)$, is given by

$$A_g(x,y) := \arccos \frac{g(x,y)}{\|x\|.\|y\|}.$$

Note that $A_g(x,y) = ½\,\pi$ if and only if $g(x,y) = 0$ or $x \perp_g y$. In an inner product space $(X,\langle\bullet,\bullet\rangle)$, $g$-angle is identical with the usual angle.

**Fact 3.3** $g$-angle satisfies the following properties:

(a) If $x$ and $y$ are of the same direction, then $A_g(x,y) = 0$; if $x$ and $y$ are of opposite direction, then $A_g(x,y) = \pi$ (part of parallelism property).

(b) $A_g(ax,by) = A_g(x,y)$ if $ab > 0$; $A_g(x,y) = \pi - A_g(x,y)$ if $ab < 0$ (homogeneity property);

(c) If $y_n \to y$ (in norm), then $A_g(x,y_n) \to A_g(x,y)$ (part of continuity property).

*Proof.*

(a) If $y = kx$ with $k > 0$, then $A_g(x,y) = \text{arcos}(1) = 0$. If $y = kx$ with $k < 0$, then $A_g(x,y) = \arccos(-1) = \pi$.

(b) If $ab > 0$, then $A_g(ax,by) = A_g(x,y)$ because the factor $ab$ is cancelled from both numerator and denominator when we compute the arccos. If $ab < 0$ and $A_g(x,y) = \arccos u(x,y)$, then $A_g(ax,by) = \arccos(-u(x,y))$, and hence $A_g(ax,by) = \pi - A_g(x,y)$.

(c) If $y_n \to y$ (in norm), then we have $g(x,y_n - y) \to 0$ because $g(x,y_n - y) \leq \|x\|.\|y_n - y\|$ and $\|y_n - y\| \to 0$. But $g(x,y_n - y) = g(x,y_n) - g(x,y)$, and so $g(x,y_n) \to g(x,y)$.

*Remark*. Since $g$ in general is not commutative, $g$-angle does not satisfy symmetry property. For example, in $l^1$ with $g(x,y) := \|x\|_1.\sum \text{sgn}(\xi_k).\eta_k$, take $x = (-1,2,0,0,0,\ldots)$ and $y = (1,1,0,0,0,\ldots)$, for which $g(x,y) = 0 \neq g(y,x)$ Continuity property also fails to hold, but we leave it to the reader to find a counter-example.

Now we move to the notion of *D*-angle. Here we suppose that $X$ is also equipped with a *2-norm* $\|\bullet,\bullet\|$ satisfying

(v) $\|x,y\| \ge 0$ for every $x$, $y$ in $X$; $\|x,y\| = 0$ if and only if $x$ and $y$ are linearly dependent;
(vi) $\|x,y\| = \|y,x\|$ for every $x$ and $y$ in $X$;
(vii) $\|ax,y\| = |a|.\|x,y\|$ for every $x$, $y$ in $X$ and $a$ in $\mathbf{R}$;
(viii) $\|x_1 + x_2,y\| \le \|x_1,y\| + \|x_2,y\|$ for every $x_1$, $x_2$, $y$ in $X$.

Geometrically, $\|x,y\|$ may be interpreted as the area of the parallelogram spanned by $x$ and $y$ in $X$. For example, in an inner product space $(X, <\bullet,\bullet>)$, the mapping

$$\|x,y\|_S := [\|x\|^2\|y\|^2 - <x,y>^2]^{1/2}$$

defines a 2-norm on $X$, called the *standard 2-norm*, which does represent the area of the parallelogram spanned by $x$ and $y$. In a normed space $(X, \|\bullet\|)$, a 2-norm $\|\bullet,\bullet\|$ may be defined by using linear functionals on $X$. For historical background of 2-norms, see [8].

With the 2-norm, we have the following notion of *D*-orthogonality on $X$:

3.4 *D-orthogonality*: $x$ is *D*-orthogonal to $y$, denoted by $x \perp_D y$, if and only if $\|x,y\| = \|x\|.\|y\|$.

If, in addition, the 2-norm also satisfies $\|x,y\| \le \|x\|.\|y\|$ for every $x$ and $y$ in $X$, then we can define *D*-angle in $X$ as follows:

3.5 *D-angle* between $x$ and $y$, denoted by $A_D(x,y)$, is given by

$$A_D(x,y) := \arcsin \frac{\|x,y\|}{\|x\|.\|y\|}.$$

Note that $A_D(x,y) = \frac{1}{2}\pi$ if and only if $\|x,y\| = \|x\|.\|y\|$, that is if and only if $x \perp_D y$. In an inner product space equipped with the standar 2-norm $\|\bullet,\bullet\|_S$, one may observe that

$$\sin A_D(x,y) = \sin A(x,y),$$

and hence $A_D(x,y) = A(x,y)$ when $A(x,y)$ is acute, or $A_D(x,y) = \pi - A(x,y)$ when $A(x,y)$ is obstuse.

**Fact 3.6** *D*-angle satisfies the following properties:

(a) $A_D(x,y) = 0$ if and only if $x$ and $y$ are linearly dependent (part of parallelism property);
(b) $A_D(x,y) = A_D(y,x)$ for every $x$, $y$ in $X$ (symmetry property);
(c) $A_D(ax,by) = A_D(x,y)$ for every $x,y$ in $X$ and $a$, $b$ in $\mathbf{R}$ (part of homogeneity property);
(d) If $x_n \rightarrow x$ and $y_n \rightarrow y$ (in norm), then $A_D(x_n,y_n) \rightarrow A_D(x,y)$ (continuity property).

*Proof*. Part (a), (b), and (c) follows from the properties of norms and the 2-norm. Part (d) follows from the continuity of norms, the 2-norm, and the arcsin. The continuity of the 2-norm can be established as follows. Suppose that $x_n \rightarrow x$ and $y_n \rightarrow y$. Since

$$\begin{aligned}\big|\|x_n - y_n\| - \|x,y\|\big| &\leq \big|\|x_n,y_n\| - \|x_n,y\|\big| + \big|\|x_n,y\| - \|x,y\|\big| \\ &\leq \|x_n, y_n - y\| + \|x_n - x, y\| \leq \|x_n\| \,.\, \|y_n - y\| + \|x_n - x\| \,.\, \|y\|,\end{aligned}$$

we have $\|x_n,y_n\| \rightarrow \|x,y\|$, as claimed.

*Remark*. Fact 3.6(a) and (c) suggests that $D$-angle can actually be used to define the angle between two lines spanned by $x$ and by $y$.

## 4 Discussion and Concluding Remarks

In an inner product space, we are equipped with the notion of angles which enables us to, among other things, approximate one vector by another in a certain subspace or compute how close a vector to a subspace. In a normed space, we only have the norm to work with. Using the norm, we can measure the distance between two vectors but not, for example, the angle between them. To discuss whether two vectors are orthogonal, for instance in order to measure their independence, several notions of orthogonality in normed spaces have been developed decades ago. Unlike in inner product spaces, these notions of orthogonality are not derived from the notions of angles. This research is trying to fix the hole by constructing some definitions of angles that are related to existing notions of orthogonality. We also discuss their properties and compare with the usual angle in an inner product space.

Related to the concept of $P$- and $I$-orthogonality, we can easily define $P$- and $I$-angle via the Cosine Law and Polarization Identity. In an inner product space, $P$- and $I$-angle coincide with the usual one. In a normed space in general, $P$- and $I$-angle have similar properties: they both satisfy parts of parallelism property, symmetry property, parts of homogeneity property, and continuity property. Compared to the usual angle in inner product spaces, however, some parts of parallelism and homogeneity properties are missing. Some examples have already been given.

In 1993, Milicic [3] introduced *g*-orthogonality in normed spaces, via Gateaux derivatives. Related to *g*-orthogonality, one has the notion of *g*-angle. In terms of formulae, *g*-angle is much more similar to the usual angle than *P*- and *I*-angle. Indeed, using the functional $g(\mathrm{x},y)$, one may do a kind of Gram-Schmidt process. Unfortunately, *g*-angle only satisfies parts of parallelism property, homogeneity property, and parts of continuity property. Here the symmetry property fails to hold because the functional $g(x,y)$ is not commutative.

A decade earlier than *g*-orthogonality, another concept of orthogonality in normed spaces was developed by Dimminie [6] by using the notion of 2-norms. In this paper, we define *D*-angle which is related to *D*-orthogonality, also by using 2-norms. Since the 2-norm $||x,y||$ represents the area of the parallelogram spanned by $x$ and $y$, the formula for *D*-angle between $x$ and $y$ is naturally derived from the Sine Law, assuming that $||x,y|| \le ||x||.||y||$. Here *D*-angle satisfies parts of parallelism property, symmetry property, parts of homogeneity property, and continuity property. Unlike the first three notions of angles, *D*-angle is more suitabe for measuring the angle between a vector and a one dimensional subspace or the angle between two one-dimensional subspaces, that is, between two lines.

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