CONIC OPTIMIZATION, WITH APPLICATIONS TO (ROBUST) TRUSS TOPOLOGY DESIGN

C. Roos[1], D. Chaerani[1,2]

[1]Faculty of Information Technology and Systems,
Delft University of Technology, The Netherlands
[2]Jurusan Matematika FMIPA, Univ. Padjajaran Bandung, Indonesia

Abstract

After a brief introduction to the field of Conic Optimization we present some interesting applications to the (robust) truss topology design (TTD) problem, where the goal is to design a truss of a given weight best able to withstand a set of given loads. We present a linear model for the single-load case and semidefinite models for the multi-load and the robust TTD problem. All models are illustrated by examples. It is also shown that by using duality the size of some of these models can be reduced significantly.

Keywords: conic optimization, truss topology design, conic quadratic optimization, semidefinite optimization, robust optimization.

1 Introduction

To make optimal decisions is one of the most basic desires of a human being. Whenever the situation and the targets admit a tractable mathematical formalization, this desire can, to some extent, be met by tools offered by the optimization theory and algorithms. A very general mathematical setting of an optimization problem is the following:

$$\min_{x \in X} \{f_0(x) \ : \ f_i(x) \leq 0, \, i = 1, \ldots, m\}. \tag{P}$$

In this problem, we are given an objective function $f_0(x)$ and finitely many functional constraints $f_i(x) \leq 0, i = 1, \ldots, m$. The functions $f_i(x)$ are real-valued functions of an n-dimensional design vector x varying in a given domain X. The goal is to minimize the objective over the feasible set of the problem, i.e., the set which is cut off the domain X by the system of inequalities $f_i(x) \leq 0, i = 1, \ldots, m$. In general, this is a very hard problem to solve. The situation is much better if all functions $f_i(x), i = 0, 1, \ldots, m$ are convex. In that case (P) is called a Convex Optimization problem. But even then, the problem might be hard to solve.

In this paper we restrict ourselves to a special class of convex optimization problem, namely Conic Optimization (CO) problems. Conic optimization addresses the problem of minimizing a linear objective function over the intersection of an affine set and a convex cone. The general form is as follows

$$\min_{x \in \mathbf{R}} \left\{ c^T x \; : \; Ax - b \in \mathcal{K} \right\}. \tag{CP}$$

The objective function is $c^T x$, with objective vector $c \in \mathbf{R}^n$. Furthermore, $Ax - b$ represents an affine function from $\mathbf{R}^n$ to $\mathbf{R}^m$ and $\mathcal{K}$ denotes a convex cone in $\mathbf{R}^m$. Usually A is given as an $m \times n$ (constraint) matrix, and $b \in \mathbf{R}^m$. The importance of this class of problems is due to two facts: first many nonlinear problems can be modelled as a conic optimization problem, and, secondly, under some weak conditions on the underlying cone $\mathcal{K}$, conic optimization problems can be solved efficiently.

The most easy and most well known case occurs when the cone $\mathcal{K}$ is the nonnegative orthant of $\mathbf{R}^m$, i.e. when $\mathcal{K} = \mathbf{R}^m_+$:

$$\min_{x \in \mathbf{R}} \left\{ c^T x \; : \; Ax - b \in \mathbf{R}^m_+ \right\}. \tag{LO}$$

This is nothing else as one of the standard forms of the well known Linear Optimization (LO) problem. Thus it becomes clear that LO is a special case of CO. It is well known that LO models cover numerous applications. Whenever applicable, LO allows to obtain useful quantitative and qualitative information on the problem at hand. The specific analytic structure of an LO problem gives rise to a number of general results which provide in many cases valuable insight and understanding. At the same time, this analytic structure underlies some specific computational techniques for LO; these techniques, which by now are perfectly well developed, allow to solve routinely quite large (with tens/hundreds of thousands of variables and constraints) LO problems. Nevertheless, there are many situations in reality which cannot be covered by LO models. To handle these "essentially nonlinear" cases, there is a strong need to extend the basic theoretical results and computational techniques known for LO beyond the bounds of LO.

When passing from a generic LO problem to its nonlinear extensions, we should expect to encounter some nonlinear components in the problem. Historically, this was done by putting the nonlinearity in the functions defining the problem, as done above in problem (P). In conic optimization, however, we replace the cone $\mathbf{R}^n_+$ in (LO) by a nonlinear convex cone $\mathcal{K}$, and hence the nonlinearity is now captured in the cone. In the next section we discuss some basic properties of relevant convex cones and we introduce two special cones that play prominent role in the context of conic optimization.

In the recent years, a lot of attention has been devoted to conic optimization. The reason is that the interior-point methods that were developed in the two last decades for LO (see, e.g., [19, 22, 24, 23]), and which revolutionized the field of LO, could be naturally extended to obtain polynomial-time methods for CO (see, e.g. [18]). This opened the way to a wide spectrum of new applications which cannot be captured by LO, e.g. in control theory, combinatorial optimization, etc. For a complete survey both of the theory of CO and its applications, we refer to the recent book [8].

The aim of the paper is to introduce the reader to the theory of CO, and to illustrate its use. LO has a beautiful duality theory. We will see that much of this theory can be generalized to CO. We deal with one of the important applications of conic optimization, namely truss topology design (TTD). A truss is a mechanical construction comprising thin elastic bars linked to each other, such as an electric mast, a railroad bridge, or the Eiffel tower. The TTD problem deals with how to design an optimal truss, with a given weight, best able to withstand a given load. The TTD problem has been studied extensively, both mathematically and algorithmically [1, 2, 5, 13, 17, 25]. The approach in this paper is mainly based on [8]; some new examples of truss designs are given in the course of the paper.

The paper is organized as follows. Section 2 introduces the theory of CO including the main duality results for CO. Section 3 is devoted to the TTD problem. A nonlinear and a linear model of the single load TTD problem are derived in Section 3.1. In this section we give three examples of truss designs. A fourth example is used to demonstrate the instability of the design with respect to additional loads, in Section 3.1.7. The same example is used in subsequent sections to show how more stable designs can be obtained.

Based on a variational principle, introduced in Section 3.2.1, we derive a model for the TTD problem that enables us to deal with the multi-load case, i.e., the case where we want to design a truss that is able to withstand a finite set of different loads in the best possible way. The model is a conic optimization model, of the semidefinite type. A simple example of a multi-load design is presented, and it is shown that the new design may be very sensitive to small occasional loads. Finally, to make the design less sensitive to such perturbations in the load, in Section 3.2.4 we make use of a recently developed modelling technique (see, e.g., [3, 4, 6, 7, 9, 10, 11, 12, 14, 15, 21]) that yields a very robust design.

2 Conic optimization

The general form of a conic optimization problem is as given by (CP). In this section we start with a discussion of the conditions on the cone $\mathcal{K}$, and we review the three most important cones. Then we deal with the main duality results for CO. It will become clear that under some mild conditions the duality theory for CO closely resembles the well known duality theory for LO.

2.1 More on convex cones

Recall that a subset $\mathcal{K}$ of $\mathbf{R}^m$ is a cone if

$$a \in \mathcal{K},\, \lambda \geq 0 \;\Rightarrow\; \lambda a \in \mathcal{K}, \tag{1}$$

and the cone $\mathcal{K}$ is a convex cone if moreover

$$a, a' \in \mathcal{K} \Rightarrow a + a' \in \mathcal{K}. \tag{2}$$

We will impose three more conditions on $\mathcal{K}$. Recall that CO is a generalization of LO. To obtain duality results for CO similar to those for LO, the cone $\mathcal{K}$ should inherit three more properties from the cone underlying LO, namely the nonnegative orthant:

$$\mathbf{R}^m_+ = \left\{ x = (x_1, \ldots, x_m)^T \;:\; x_i \geq 0,\, i = 1, \ldots, m \right\}.$$

This cone is called the *linear cone*. The linear cone is not just a convex cone; it is also pointed, it is closed and it has a nonempty interior. These are exactly the three properties we need. We describe these properties now. A convex cone $\mathcal{K}$ is called pointed if it does not contain a line. This property can be stated equivalently as

$$a \in \mathcal{K},\; -a \in \mathcal{K} \Rightarrow a = 0. \tag{3}$$

A convex cone $\mathcal{K}$ is called closed if it is closed under taking limits:

$$a_i \in \mathcal{K}\ (i = 1, 2, \ldots),\; a = \lim_{i \to \infty} a_i \Rightarrow a \in \mathcal{K}. \tag{4}$$

Finally, denoting the interior of a cone $\mathcal{K}$ as $\operatorname{int} \mathcal{K}$, we will require that

$$\operatorname{int} \mathcal{K} \neq \emptyset. \tag{5}$$

This means that there exists a vector (in $\mathcal{K}$) such that a ball of positive radius centered at the vector is contained in $\mathcal{K}$. In conic optimization we only deal with cones $\mathcal{K}$ that enjoy all of the above properties. So we always assume that $\mathcal{K}$ is *a pointed and closed convex cone with a nonempty interior.* Apart from the linear cone, two other relevant examples of such cones are

1. The *Lorentz cone*

   $$\mathbf{L}^m = \{x \in \mathbf{R}^m \;:\; x_m \geq \sqrt{x_1^2 + \ldots + x_{m-1}^2}\}.$$

   This cone is also called the *second-order cone*, or the *ice-cream cone*.

2. The *positive semidefinite cone* $\mathbf{S}^m_+$. This cone "lives" in the space $\mathbf{S}^m$ of $m \times m$ symmetric matrices (equipped with the Frobenius inner product $\langle A, B \rangle = Tr(AB) = \sum_{i,j} A_{ij}B_{ij}$) and consist of all $m \times m$ matrices A which are positive semidefinite, i.e.,

$$\mathbf{S}^m_+ = \left\{ A \in \mathbf{S}^m \; : \; x^T Ax \geq 0, \quad \forall x \in \mathbf{R}^m \right\}.$$

We assume that the cone $\mathcal{K}$ in (CP) is a direct product of the form

$$\mathcal{K} = \mathcal{K}^1 \times \ldots \times \mathcal{K}^m,$$

where each component $\mathcal{K}^i$ is either a linear, a Lorentz or a semidefinite cone.

2.2 Conic Duality

Before we derive the duality theory for conic optimization, we need to define the *dual cone* of a convex cone $\mathcal{K}$:

$$\mathcal{K}_* = \left\{ \lambda \in \mathbf{R}^m \; : \; \lambda^T a \geq 0, \, \forall a \in \mathcal{K} \right\}. \tag{6}$$

We recall the following result from [8].

Theorem 2.1 *Let $\mathcal{K} \subset \mathbf{R}^m$ is a nonempty cone. Then*

(*i*) *The set $\mathcal{K}_*$ is a closed convex cone.*

(*ii*) *If $\mathcal{K}$ has a nonempty interior (i.e., $\operatorname{int} \mathcal{K} \neq \emptyset$) then $\mathcal{K}_*$ is pointed.*

(*iii*) *If $\mathcal{K}$ is a closed convex pointed cone, then $\operatorname{int} \mathcal{K}_* \neq \emptyset$.*

(*iv*) *If $\mathcal{K}$ is a closed convex cone, then so is $\mathcal{K}_*$, and the cone dual to $\mathcal{K}_*$ is $\mathcal{K}$ itself.*

Corollary 2.2 *If $\mathcal{K} \subset \mathbf{R}^m$ is a closed pointed convex cone with nonempty interior then so is $\mathcal{K}_*$, and vice versa.*

One may easily verify that the three cones introduced in Section 2.1 are self-dual. The dual of a direct product of convex cones is the direct product of their duals, i.e.,

$$\mathcal{K} = \mathcal{K}^1 \times \ldots \times \mathcal{K}^m \quad \Rightarrow \quad \mathcal{K}_* = \mathcal{K}^1_* \times \ldots \times \mathcal{K}^m_*.$$

As a consequence, any direct product of linear, Lorentz and semidefinite cones is self-dual.

Now we are ready to deal with the problem dual to a conic problem (CP). We start with observing that whenever x is a feasible solution for (CP) then the definition of $\mathcal{K}_*$ implies $\lambda^T(Ax-b) \geq 0$, for all $\lambda \in \mathcal{K}_*$, and hence x satisfies the scalar inequality

$$\lambda^T Ax \geq \lambda^T b, \quad \forall \lambda \in \mathcal{K}_*.$$

It follows that whenever $\lambda \in \mathcal{K}_*$ satisfies the relation

$$A^T\lambda = c \tag{7}$$

then one has

$$c^T x = (A^T\lambda)^T x = \lambda^T Ax \geq \lambda^T b = b^T\lambda$$

for all x feasible for (CP). So, if $\lambda \in \mathcal{K}_*$ satisfies (7), then the quantity $b^T\lambda$ is a lower bound for the optimal value of (CP). The best lower bound obtainable in this way is the optimal value of the problem

$$\max_{\lambda \in \mathbf{R}} \left\{ b^T\lambda \ : \ A^T\lambda = c, \ \lambda \in \mathcal{K}_* \right\}. \tag{CD}$$

By definition, (CD) is the *dual problem* of (CP). Using Theorem 2.1 (iv), one easily verifies that the duality is symmetric: the dual problem is conic and the problem dual to the dual problem is the primal problem.

Indeed, from the construction of the dual problem it immediately follows that we have the weak duality property: if x is feasible for (CP) and λ is feasible for (CD), then

$$c^T x - b^T\lambda \geq 0.$$

The crucial question is, of course, if we have equality of the optimal values whenever (CP) and (CD) have optimal values. Different from the LO case, however, this is in general not the case, unless some additional conditions are satisfied. The following theorem clarifies the situation. For its proof we refer again to [8]. We call the problem (CP) *solvable* if it has a (finite) optimal value, and this value is attained. Before stating the theorem it may be worth pointing out that a finite optimal value is not necessarily attained. For example, the problem

$$\min_x \left\{ x \ : \ \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} \succeq 0 \right\}$$

has optimal value 0, but one may easily verify that this value is not attained. We need one more definition: if there exists an x such that $Ax - b \in \operatorname{int}\mathcal{K}$ then we say that (CP) is *strictly feasible*. We have similar, and obvious, definitions for (CD) being solvable and strictly feasible, respectively.

Theorem 2.3 *Let the primal problem (CP) and its dual problem (CD) be as given above. Then one has*

(i) a. *If (CP) is below bounded and strictly feasible, then (CD) is solvable and the respective optimal values are equal.*

b. If (CD) is above bounded and strictly feasible, then (CP) is solvable, and the respective optimal values are equal.

(ii) Suppose that at least one of the two problems (CP) and (CD) is bounded and strictly feasible. Then a primal-dual feasible pair (x, λ) is comprised of optimal solutions to the respective problems

a. if and only if $b^T\lambda = c^T x$ (zero duality gap).

b. if and only if $\lambda^T[Ax - b] = 0$ (complementary slackness).

Note that this result is slightly weaker than the corresponding result for the LO case. In the LO case the same theorem holds by putting everywhere "feasible" instead of "strictly feasible". The adjective "strictly" cannot be omitted here, however. For a more extensive discussion and some appropriate counterexamples we refer to [8].

3 The truss topology design problem

A truss is a mechanical construction comprising thin elastic bars linked to each other, such as an electric mast, a railroad bridge, or the Eiffel tower. The points at which the bars are linked to each other are called the nodes of the truss. A truss can be subjected to an external load -- a collection of simultaneous forces acting at the nodes, as shown by example in Figure 1. Some nodes of the truss are fixed nodes (like the nodes A, B and A' in the figure), whereas the remaining nodes are called free nodes. In some of the free nodes (nodes C, C' and E in the figure) an external load acts on the truss.

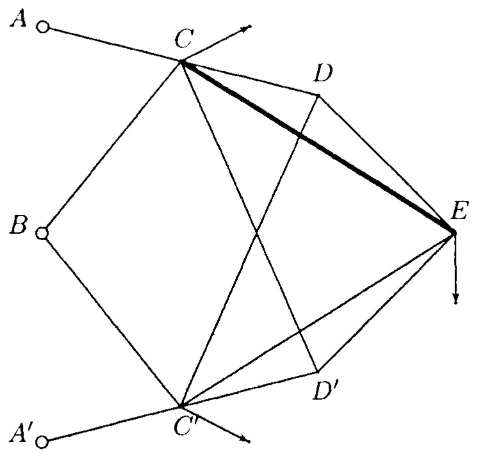

Figure 1: A simple planar truss with a load

Under the load, the truss deforms a bit, until the forces in the bars caused by the deformation, compensate the external forces. When deformed, the truss stores certain potential energy; this energy is called the compliance of the truss with respect to the load. The less the compliance, the more rigid the truss is with respect to the load in question.

Our goal is to design a truss of a given total weight best able to withstand a given load. We call this the TTD problem.

This section is organized as follows. First we derive a linear model for the single load case and give some examples. ¿From the examples it becomes clear that the solution obtained from the linear model can be very instable with respect to occasional small additional loads. This makes it necessary to consider the case of multi-loads. This case cannot be modelled in a linear way, but we can do it by using a semidefinite model. To makes such a model robust against arbitrary perturbations (of limited size) in the load we need another semidefinite model.

3.1 A nonlinear and a linear model of the single-load TTD problem

3.1.1 Force and potential energy in a single bar

To start with, let us look in more detail at what happens with a bar in the truss, due to the displacements of the nodes in the truss when the external forces are working. Consider a particular bar AB in the unloaded truss. Let ΔA and ΔB denote the displacements of the nodes A and B. Defining

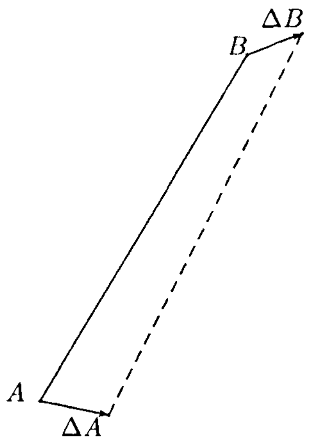

Figure 2: A bar before (solid) and after (dashed) load is applied.

$$x = B - A, \quad \Delta x = \Delta B - \Delta A,$$

and assuming that ΔA and ΔB are small relative to $\|AB\| = \ell$, a first order approximation of the elongation $\Delta \ell$ of the bar is given by

$$\Delta \ell = \frac{x^T \Delta x}{\|x\|}. \tag{8}$$

Let ℓ denote the length of the bar, so $\ell = \|x\|$. The tension caused by the elongation is given by Hooke's law:

$$\sigma = \kappa \frac{\Delta \ell}{\ell} = \kappa \frac{x^T \Delta x}{\|x\|^2}, \tag{9}$$

where κ is a characteristic of the material (known as Young's modulus). Hence, if S_{AB} denotes the surface of a cross-section of the bar, then the magnitude of the force caused by the elongation, is given by

$$|F| = \sigma \times S_{AB} = \sigma \frac{t_{AB}}{\ell} = \kappa \, t_{AB} \, \frac{x^T \Delta x}{\|x\|^3}, \tag{10}$$

where t_{AB} denotes the volume of the bar. Thus, the force caused by the bar at node A is given by

$$F = |F| \frac{x}{\|x\|} = \kappa \, t_{AB} \, \frac{x^T \Delta x}{\|x\|^4} \, x, \tag{11}$$

and the force at node B is $-F$.

It will be convenient to associate the vector

$$\beta_{AB} = \sqrt{\kappa} \, \frac{x}{\|x\|^2}$$

to the bar AB. Note that, given κ, the vector β_{AB} contains information both on the direction of the bar AB (the same as β_{AB}) and the length of the bar, since

$$\|\beta_{AB}\| = \frac{\sqrt{\kappa}}{\|x\|}.$$

The tension in the bar can then be written as

$$\sigma = \kappa \frac{x^T \Delta x}{\|x\|^2} = \sqrt{\kappa} \, \Delta x^T \beta_{AB}, \tag{12}$$

and the force at A caused by bar AB is given by

$$F = \kappa \, t_{AB} \, \frac{x^T \Delta x}{\|x\|^4} \, x = t_{AB} \left(\Delta x^T \beta_{AB} \right) \beta_{AB}. \tag{13}$$

Now we can deal with the potential energy stored in the bar as a result of its elongation. ¿From mechanics we know that this energy is given by[1]

$$\tfrac{1}{2} |F| \times \Delta \ell = \tfrac{1}{2} \Delta x^T F = \tfrac{1}{2} t_{AB} \left(\Delta x^T \beta_{AB} \right)^2. \tag{14}$$

[1]Let ξ be the elongation of the bar ($0 \le \xi \le \Delta\ell$). The force necessary to maintain this elongation is given by $\gamma\xi$ for some material constant γ. When increasing the elongation with $d\xi$ the additional amount of energy stored in the bar is $\gamma\xi d\xi$. Hence, when reaching the elongation $\Delta\ell$, the total potential energy stored in the bar is given by

$$\int_0^{\Delta\ell} \gamma\xi d\xi = \tfrac{1}{2}\gamma\xi^2 \,\big|_0^{\Delta\ell} = \tfrac{1}{2}\gamma(\Delta\ell)^2 = \tfrac{1}{2} F \Delta\ell,$$

where F denotes the force corresponding to the elongation $\Delta\ell$.

3.1.2 Forces and potential energy in the whole truss

Let V (V_f) denote the set of (free) nodes in the truss. For each node $v \in V$, Δv will denote the displacement of v. Note that $v \in \mathbf{R}^2$ if the truss is planar, and $v \in \mathbf{R}^3$ if the truss is spatial. Below we assume that $v \in \mathbf{R}^d$, with $d \in \{2, 3\}$. If v is a fixed node then $\Delta v = 0$, so Δv can be nonzero only if v is a free node.

We denote by ΔV the concatenation of the vectors Δv in the free nodes, and use the notation $\Delta V(v) = \Delta v$. Then $\Delta v \in \mathbf{R}^d$ and $\Delta V \in \mathbf{R}^{d|V_f|}$. The external load is considered to be a vector in the same space: $f \in \mathbf{R}^{d|V_f|}$; so $f(v)$ denotes the external force in the free node v.

It will be convenient to consider a bar as an ordered pair of nodes; thus we assign a direction to each bar (in an arbitrary way). Then any bar has the form (v, w), where $v, w \in V$. The vector $\beta_{vw} \in \mathbf{R}^d$ is then given by

$$\beta_{vw} = \sqrt{\kappa}\,\frac{w - v}{\|w - v\|^2}, \tag{15}$$

and, by (13), the force realized by bar (v, w) at node v is given by

$$t_{vw}\left(\left(\Delta w - \Delta v\right)^T \beta_{vw}\right)\beta_{vw} = t_{vw}\left(\Delta w^T \beta_{vw} + \Delta v^T \beta_{wv}\right)\beta_{vw}.$$

Let $\mathcal{A}$ denote the set of all bars. For each bar $a = (v, w) \in \mathcal{A}$ we define a vector $b_{vw} \in \mathbf{R}^{d|V_f|}$ according to

$$b_{vw}(u) = \begin{cases} \beta_{vw}, & \text{if } u = w, \\ -\beta_{vw} = \beta_{wv}, & \text{if } u = v, \\ 0 & \text{otherwise}. \end{cases} \qquad u \in V_f. \tag{16}$$

Then the force realized by bar (v, w) at node v can be written as

$$\begin{aligned} t_{vw}\left(\Delta w^T \beta_{vw} + \Delta v^T \beta_{wv}\right)\beta_{vw} &= t_{vw}\left(\Delta w^T b_{vw}(w) + \Delta v^T b_{vw}(v)\right)\beta_{vw} \\ &= t_{vw}\left(\Delta V^T b_{vw}\right)\beta_{vw} \\ &= t_{vw}\left(\Delta V^T b_{vw}\right) b_{vw}(v). \end{aligned}$$

Consequently, the total force at node v, caused by the elongations of the bars connected to v, will be given by

$$\sum_{\{v,w\} \in \mathcal{A}} t_{vw}\left(\Delta V^T b_{vw}\right) b_{vw}(v) = \left(\sum_{\{v,w\} \in \mathcal{A}} t_{vw} b_{vw}(v) b_{vw}^T\right)\Delta V,$$

where $\{v, w\} \in \mathcal{A}$ is a short-hand notation for either $(v, w) \in \mathcal{A}$ or $(w, v) \in \mathcal{A}$. The above expression is a vector in $\mathbf{R}^d$, as it should. By concatenation of all these vectors we get the vector

$$\sum_{\{v,w\} \in \mathcal{A}} t_{vw}\left(\Delta V^T b_{vw}\right) b_{vw} = \left(\sum_{\{v,w\} \in \mathcal{A}} t_{vw} b_{vw} b_{vw}^T\right)\Delta V.$$

This vector represents the forces acting from within the truss on its free nodes. In equilibrium, these forces have to compensate the external forces acting at the free nodes. This implies that the following equation should be satisfied:

$$\left(\sum_{\{v,w\}\in\mathcal{A}} t_{vw} b_{vw} b_{vw}^T\right) \Delta V = f.$$

The matrix

$$A(t) = \sum_{\{v,w\}\in\mathcal{A}} t_{vw} b_{vw} b_{vw}^T \tag{17}$$

is called the *bar-stiffness matrix* of the truss. This is an $|\mathcal{A}| \times |\mathcal{A}|$ matrix. Note that the bar-stiffness matrix depends linearly on the volumes of the bars. We conclude that in equilibrium the displacements vector ΔV will satisfy the following linear system of equations.

$$A(t)\,\Delta V = f. \tag{18}$$

Remark 3.1 *The $|V_f| \times |\mathcal{A}|$ matrix B whose columns are the vectors b_{vw} is called the node-bar matrix of the truss. This matrix depends on the truss alone, i.e., on the bars and their volumes. Note that we have the following simple relation between the node-bar matrix and the bar-stiffness matrix:*

$$A(t) = B\,\mathrm{diag}\,(t)\,B^T. \tag{19}$$

To complete our model, we should also find an expression for the compliance, i.e. the potential energy stored in the truss. According to (14) and (16) this potential energy is given by

$$\mathrm{Compl}_f(t) = \tfrac{1}{2} \sum_{\{v,w\}\in\mathcal{A}} t_{vw} \left(\Delta V^T b_{vw}\right)^2. \tag{20}$$

Using (17) and (18), this can be reduced as follows:

$$\begin{aligned}\mathrm{Compl}_f(t) &= \tfrac{1}{2}\,\Delta V^T \left(\sum_{\{v,w\}\in\mathcal{A}} t_{vw} b_{vw} b_{vw}^T\right) \Delta V \\ &= \tfrac{1}{2}\,\Delta V^T A(t) \Delta V = \tfrac{1}{2}\, f^T \Delta V. \end{aligned} \tag{21}$$

Remark 3.2 *From a physical point of view it seems to be clear that* $\mathrm{Compl}_f(t)$ *depends on t and f only, and not on ΔV. From a mathematical point of view, however, this is not evident, since the equilibrium equation (18) may have more than one solution (or no solution at all). If (18) has no solution, then this means that the truss t cannot carry the load f: it is crushed by this load. In this case it makes sense to define* $\mathrm{Compl}_f(t) = \infty$. *Now suppose that (18) has two solutions: x_1 and x_2. So, $A(t)x_1 = A(t)x_2 = f$. Let $x = x_1 - x_2$. Then $A(t)x = 0$. Letting $T = \mathrm{diag}\,(t)$, we thus have $BTB^Tx = 0$. This implies $x^T BTB^T x = 0$, whence $\|T^{\frac{1}{2}} B^T x\| = 0$. Hence we have $b_{vw}^T x = 0$ whenever $t_{vw} > 0$. In other words, if $t_{vw} > 0$ then $b_{vw}^T x_1 = b_{vw}^T x_2$. Therefore,*

$$\sum_{\{v,w\}\in\mathcal{A}} t_{vw} \left(x_1^T b_{vw}\right)^2 = \sum_{\{v,w\}\in\mathcal{A}} t_{vw} \left(x_2^T b_{vw}\right)^2.$$

Because of (20) this proves the claim.

by (26). One has

$$
\begin{aligned}
f^T \Delta V &\overset{(24)}{=} \sum_{\{v,w\}\in\mathcal{A}} q^*_{vw} b^T_{vw} \Delta V \\
&= \frac{\|q^*\|_1}{\omega} \sum_{\{v,w\}\in\mathcal{A}} q^*_{vw} b^T_{vw} \Delta V^* = \frac{\|q^*\|_1}{\omega} \sum_{\{v,w\}\in\mathcal{A}} |q_{vw}| = \frac{\|q^*\|_1^2}{\omega}.
\end{aligned}
$$

Thus it remains to show that if t and ΔV are arbitrary feasible solutions to (22), then

$$
f^T \Delta V \geq \frac{\|q^*\|_1^2}{\omega}, \tag{27}
$$

where q^* is any optimal solution of (24). To show this we introduce variables q_{vw} as follows:

$$
q_{vw} := t_{vw}\left(\Delta V^T b_{vw}\right), \quad \forall (v,w) \in \mathcal{A}.
$$

The vector $q = (q_{vw})_{(v,w)\in\mathcal{A}}$ is feasible for (24), because

$$
\begin{aligned}
\sum_{\{v,w\}\in\mathcal{A}} q_{vw} b_{vw} &= \sum_{\{v,w\}\in\mathcal{A}} t_{vw}\left(\Delta V^T b_{vw}\right) b_{vw} \\
&= \left(\sum_{\{v,w\}\in\mathcal{A}} t_{vw} b_{vw} b^T_{vw}\right) \Delta V = A(t)\Delta V = f.
\end{aligned}
$$

Since q^* is optimal for (24), we conclude from this that

$$
\|q\|_1 \geq \|q^*\|_1. \tag{28}
$$

The last step in this analysis consists of showing that (28) implies (27). From (20), (21) and the definition of q_{vw} we deduce that

$$
f^T \Delta V = \sum_{\{v,w\}\in\mathcal{A}} t_{vw}\left(\Delta V^T b_{vw}\right)^2 = \sum_{t_{vw}>0} \frac{q^2_{vw}}{t_{vw}}.
$$

Using the Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
\|q\|_1^2 &= \left(\sum_{t_{vw}>0} |q_{vw}|\right)^2 = \left(\sum_{t_{vw}>0} \left(t_{vw}^{-\frac{1}{2}} |q_{vw}|\right) t_{vw}^{\frac{1}{2}}\right)^2 \\
&\leq \left(\sum_{t_{vw}>0} \frac{q^2_{vw}}{t_{vw}}\right)\left(\sum_{t_{vw}>0} t_{vw}\right) \leq \omega \sum_{t_{vw}>0} \frac{q^2_{vw}}{t_{vw}}.
\end{aligned}
$$

The last inequality follows since t is feasible for (22). Since $\omega > 0$, we obtain, also using (28),

$$
\sum_{t_{vw}>0} \frac{q^2_{vw}}{t_{vw}} \geq \frac{\|q\|_1^2}{\omega} \geq \frac{\|q^*\|_1^2}{\omega},
$$

proving (27). Thus we have shown that optimal solutions of the linear problem (23) and its dual (24) contain all the information we need to obtain an optimal solution of the nonlinear problem (22).

Obviously, the dual problem is feasible if and only if the vector f of external forces belongs to the span of the vectors b_{vw}. We will make this natural assumption. The primal problem is feasible as well (take $\Delta V = 0$). Hence, by the duality theorem for linear optimization, both problems have optimal solutions and their optimal values are equal. If ΔV is primal feasible and q dual feasible, then we may write

$$\begin{aligned} f^T \Delta V = \sum_{\{v,w\} \in \mathcal{A}} q_{vw} \Delta V^T b_{vw} &\leq \sum_{\{v,w\} \in \mathcal{A}} |q_{vw}| \left| \Delta V^T b_{vw} \right| \\ &\leq \sum_{\{v,w\} \in \mathcal{A}} |q_{vw}| = \|q\|_1 . \end{aligned}$$

Hence, ΔV and q are optimal, for (23) and (24) respectively, if and only if the above two inequalities hold with equality, and this holds if and only if

$$q_{vw} \Delta V^T b_{vw} = |q_{vw}| , \quad \forall (v,w) \in \mathcal{A}. \tag{25}$$

It is worth pointing out an interesting consequence of this result, namely that if ΔV is primal feasible and q dual feasible then these solutions are optimal if and only if:

> *for every bar $\{v, w\}$ with $q_{vw} \neq 0$ one has $\left|\Delta V^T b_{vw}\right| = 1$; in other words, all such bars have the same tension, namely 1!*

3.1.5 Correctness of the linear model

In this section we show that if ΔV^* and q^* satisfy (25) (i.e., are optimal for (23) and (24) respectively) then

$$t = \omega \frac{|q^*|}{\|q^*\|_1}, \quad \Delta V = \frac{\|q^*\|_1}{\omega} \Delta V^* \tag{26}$$

is an optimal solution to our original quadratic model (22). Obviously, the nonnegativity constraint $t \geq 0$ and the volume constraint in (22) are satisfied:

$$\sum_{\{v,w\} \in \mathcal{A}} t_{vw} = \omega \frac{\sum_{\{v,w\} \in \mathcal{A}} |q^*_{vw}|}{\|q^*\|_1} = \omega.$$

Indeed, the volume constraint is tight! Furthermore, using (17) (i.e., the definition of $A(t)$), (25) and (26) we may write

$$\begin{aligned} A(t)\,\Delta V &= \left(\sum_{\{v,w\} \in \mathcal{A}} t_{vw} b_{vw} b_{vw}^T \right) \Delta V \\ &\overset{(26)}{=} \sum_{\{v,w\} \in \mathcal{A}} |q^*_{vw}|\, b_{vw} \left(b_{vw}^T \Delta V^* \right) = \sum_{q_{vw} \neq 0} |q^*_{vw}|\, b_{vw} \left(b_{vw}^T \Delta V^* \right) \\ &\overset{(25)}{=} \sum_{q_{vw} \neq 0} |q^*_{vw}|\, b_{vw} \frac{|q^*_{vw}|}{q^*_{vw}} = \sum_{q_{vw} \neq 0} b_{vw} q^*_{vw} \overset{(24)}{=} f. \end{aligned}$$

Thus we have shown that (26) is feasible for (22). It remains to show that the objective value $f^T \Delta V$ is minimal. We start with computing this value for the solution given becomes:

	b_{12}	b_{14}	b_{15}	b_{16}	b_{23}	b_{24}	b_{25}	b_{26}	b_{34}	b_{35}	b_{36}	b_{45}	b_{56}
1	−20	0	−4	−5									
	0	−10	−8	−5									
2	20				−20	4	0	−4					
	0				0	−8	−10	−8					
3					20				5	4	0		
					0				−5	−8	−10		
4		0				−4			−5			−20	
		10				8			5			0	
5			4				0			−4		20	−20
			8				10			8		0	0
6				5				4			0		20
				5				8			10		0

The non-indicated entries are all zero. By removing the rows corresponding to the fixed nodes (i.e., node 1 and node 3), we get the matrix B as given below,

$B =$

	b_{12}	b_{14}	b_{15}	b_{16}	b_{23}	b_{24}	b_{25}	b_{26}	b_{34}	b_{35}	b_{36}	b_{45}	b_{56}
2	20				-20	4	0	-4					
	0				0	-8	-10	-8					
4		0				-4			-5			-20	
		10				8			5			0	
5			4				0			-4		20	-20
			8				10			8		0	0
6				5				4			0		20
				5				8			10		0

where we have also depicted the vector f as given below,

$$f = \begin{bmatrix} 0 \\ -10 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{bmatrix}.$$

With B and f as just given, and with $w = 1$, the solution of the problem (24) is as follows:

$$q = \left(0, 0, -\frac{5}{8}, 0, 0, 0, 1, 0, 0, -\frac{5}{8}, 0, 0, 0\right)^T$$

and the solution of (23) is given by

$$\Delta V = (0, -0.225, 0.027, -0.091, 0, -0.125, -0.027, -0.092)^T.$$

Using (26) we construct the optimal solution of the nonlinear model, which gives (with $w = 1$):

$$t = \left(0, 0, \frac{5}{18}, 0, 0, 0, \frac{8}{18}, 0, 0, \frac{5}{18}, 0, 0, 0\right)$$

3.1.6 Three examples

Vertical load on two parallel walls Let us start by considering the following simple example where one has two parallel walls, with the same height and at distance 1 meter from each other. One wants to make make a vertical (flat, i.e., 2-dimensional) construction on the wall that should carry a load of 10 Newton. See Figure 3. For that purpose a fixed amount of 1 unit of material is available. To obtain the stiffest

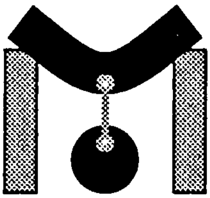

Figure 3: Two walls that need to carry the load as indicated .

possible construction, let us use a equidistant grid of size 3×2 of height 1 meter, whose vertices are then given by $(\frac{i}{2}, j)$, $i = 0, 1, 2$ and $j = 0, 1$. See Figure 4. The

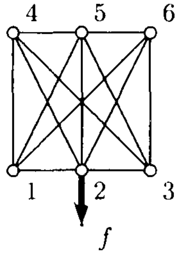

Figure 4: A 3×2 grid with external force in node 2.

nodes are numbered as indicated.

By way of example, let us compute vector β_{12}, assuming $\kappa = 100$. This bar's end nodes are 1 and 2, whose coordinates are $(0,0)$ and $(\frac{1}{2},0)$ respectively. Since the length of this bar is $1/2$, using (15) we find

$$\beta_{12} = \sqrt{100} \times \frac{1}{\left(\frac{1}{2}\right)^2} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 20 \\ 0 \end{pmatrix}.$$

We can form the vector b_{12}, as defined by (16). For the moment we assume that all nodes are free! Then b_{12} is the first vector in the scheme below. Note that node 1 is fixed, hence, according to the definition in (16), the corresponding (first two) entries of b_{12} should be removed. We will deal with this later, simply by removing all entries in the rows corresponding to fixed nodes.

Assuming that all nodes are free, the matrix consisting of all column vectors b_{ij}

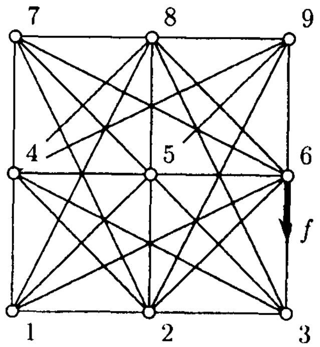

Figure 6: A 3 × 3 grid with external force in node 6.

Using (26) we construct the optimal solution of the nonlinear model, which gives (with $w = 1$):

$$t = \left(0, 0, 0, \frac{1}{2}, 0, 0, 0, \cdots, 0, 0, \frac{1}{2}, 0, 0, 0, 0\right)^T$$

and

$$\Delta V = -(\,0.1242, 0.2777, 0.2091, 0.5351, 0, 0.1946,\\ 0, 0.6250, -0.1246, 0.2775, -0.2093, 0.5352)^T.$$

¿From this we can compute the compliance:

$$\mathrm{Compl}_f(t) = \tfrac{1}{2} f^T \Delta V = \tfrac{1}{2} \times (-10) \times (-0.6250) = 3.1250.$$

The corresponding truss is shown left in Figure 7. Among trusses based on an $p \times q$

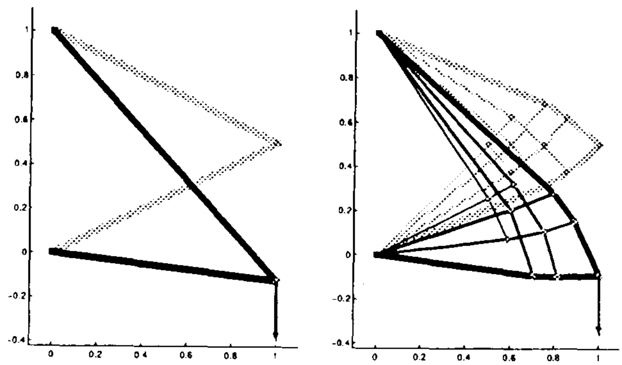

Figure 7: Trusses based on (a) 3 × 3 and (b) 21 × 17 grid.

grid, with $2 \leq p \leq 21$ and $3 \leq q \leq 21$ (q odd), the 21 × 17 grid gives the lowest compliance, namely 2.9594. The corresponding truss is shown at the right in Figure 5. The next best grid is 19 × 19, with compliance 2.9599.

and

$$\Delta V = (0, -0.5062, 0.0597, -0.2049, 0, -0.2812, -0.0615, -0.2068)^T.$$

¿From this we can compute the compliance:

$$\text{Compl}_f(t) = \tfrac{1}{2} f^T \Delta V = \tfrac{1}{2} \times (-10) \times (-0.5062) = 2.5312.$$

The corresponding truss is shown left in Figure 5. It may be clear that lower compli-

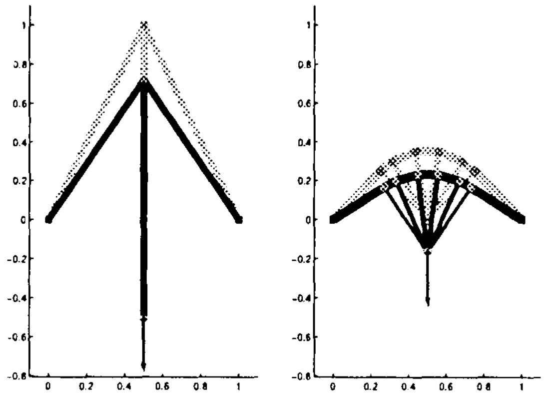

Figure 5: Trusses based on (a) 3×2 and (b) 17×21 grid.

ances may be expected when using a finer grid. For example, when using a 17×21 grid the compliance is 0.8359; the corresponding truss is shown at the right in Figure 5. Among trusses based on an $p \times q$ grid, with $3 \leq p \leq 21$ (p odd) and $2 \leq q \leq 21$, the 17×21 grid gives the lowest compliance. The next best grid is 17×20, with compliance 0.8376.

Vertical load at one vertical wall In our second example we consider the situation where a vertical load has to be carried by one vertical wall, by using a (flat 2-dimensional) truss that is fixed to one side of the wall. As before, we assume that the height and the width of the truss are 1 meter, that the acting force is 10 Newton and that an amount of 1 unit of material is available. See Figure 6, which shows an equidistant 3×3 grid for this problem; the vertices are given by $(\frac{i}{2}, \frac{j}{2})$, $i = 0, 1, 2$ and $j = 0, 1, 2$. The nodes are numbered as indicated. The fixed nodes are numbered 1, 4 and 7, and the force is acting in node 6, as indicated. The solution of the problem (24) is as follows:

$$q = \left(0, 0, 0, -\frac{5}{4}, 0, 0, 0, \cdots, 0, 0, \frac{5}{4}, 0, 0, 0, 0\right)^T$$

and the solution of (23) is given by

$$\Delta V = -(\, 0.0497, 0.1111, 0.0836, 0.2140, 0, 0.0778, \\ 0, 0.2500, -0.0498, 0.1110, -0.0837, 0.2141)^T.$$

given by

$$B = \begin{array}{c} \\ 2 \\ \\ 4 \\ \\ \end{array} \left[\begin{array}{cccccc} b_{12} & b_{13} & b_{14} & b_{23} & b_{24} & b_{34} \\ 10 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & -5 & -10 & 0 \\ \hline 0 & 0 & 5 & 0 & 0 & 10 \\ 0 & 0 & 5 & 0 & 10 & 0 \end{array}\right]; \quad f_1 = \left[\begin{array}{c} 10 \\ -0.005 \\ \hline 10 \\ 0.010 \end{array}\right].$$

Note that if the two vertical components in f_1 were zero, then an optimal truss would only use the two horizontal bars, both with half of the total volume w. But such a truss would not be able to carry the vertical components of the forces in the right nodes, and it would crash. Mathematically this means that then the problem (24) is infeasible, and problem (23) unbounded. In the present case the optimal solution of these problems are respectively given by

$$q^* = \begin{pmatrix} 1 \\ 0 \\ 0.0010 \\ 0 \\ 0.0005 \\ 0.9995 \end{pmatrix}, \qquad \Delta V^* = \begin{pmatrix} 0.1 \\ 0 \\ \hline 0.1 \\ 0.1 \end{pmatrix}.$$

Using (26) we derive from this the optimal solution of (22), with $w = 1$ and $\|q^*\|_1 = 2.001$, which gives

$$t = \frac{1}{2.001} \begin{pmatrix} 1 \\ 0 \\ 0.0010 \\ 0 \\ 0.0005 \\ 0.9995 \end{pmatrix}, \qquad \Delta V = 2.001 \begin{pmatrix} 0.1 \\ 0 \\ \hline 0.1 \\ 0.1 \end{pmatrix}$$

and $\mathrm{Compl}_{f_1}(t) = \frac{1}{2} f_1^T \Delta V = 2.0020005$. Figure 10 shows the optimal truss in the unloaded and loaded case, respectively.

The optimal truss is designed for the given load. One may wonder how it behaves when it is loaded with an occasional load that differs not too much from the design load. We will demonstrate below that very small perturbations in the load may have a disastrous effect on the compliance. To make this clear, let g be an eigenvector of $A(t)$ with eigenvalue λ. So we have $A(t)g = \lambda g$. Let f denote the design load of the truss and ΔV the corresponding displacement. Without loss of generality we assume that $\|g\| = \|f\|$ and $f^T g \geq 0$. Now consider the situation that the design load is replaced by

$$f(\gamma) = f + \gamma g$$

for some $\gamma \geq 0$. Then the displacement $\Delta V(\gamma)$ under the load follows by solving the equation $A(t)\Delta V(\gamma) = f + \gamma g$, which gives

$$\Delta V(\gamma) = \Delta V + \frac{\gamma}{\lambda} g.$$

Uniformly loaded bridge In our third example we want to design a bridge that crosses a river. The bridge rests on the shores of the river and it has to carry a load of 10 N, which is assumed to be uniformly distributed over the whole bridge. We use a $p \times q$ grid, with the lower left and lower right node fixed, and with external forces of size $10/(q-2)$ in the remaining lower nodes of the grid. As before, we assume that the height and the width of the truss are 1 meter, and that an amount of 1 unit of material is available. Since the solution t of problem is proportional to the amount of material the sizes of the bars in the truss can be easily adapted to more realistic values. A similar argument applies to the chosen size of the forces acting on the bridge and to the size of the grid.

For a 6×21 grid the corresponding solution is shown left in Figure 8. Its compliance is 0.5000. Among trusses based on an $p \times q$ grid, with $3 \leq p \leq 21$ and $3 \leq q \leq 21$ (q

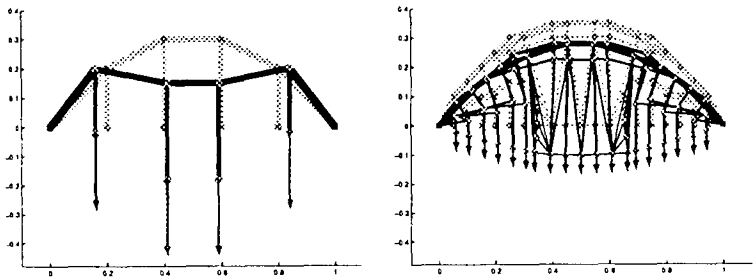

Figure 8: Trusses based on (a) 6×21 and (b) 21×21 grid.

odd), the 21×21 grid gives the lowest compliance, namely 0.3601. The corresponding truss is shown at the right in Figure 8.

3.1.7 Instability with respect to additional loads

By way of example we consider a very simple truss, based on a 2×2 square grid, for which the left nodes are fixed, and with external forces in the two right nodes. The two external forces have a horizontal component 10, and vertical components 0.005 and -0.005 in the upper and lower right node respectively. The situation is depicted in Figure 9. The nodes are numbered as indicated. The matrix B (for $\kappa = 100$) and

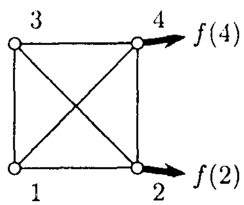

Figure 9: A 2×2 grid with external forces in nodes 2 and 4.

the vector $f = f_1$, with the entries corresponding to the fixed nodes removed, are

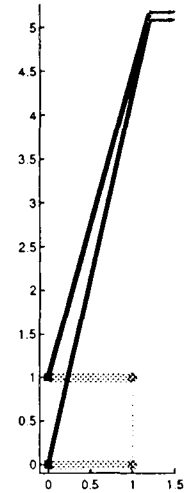

Figure 11: Instability of the truss w.r.t small occasional load.

which amounts to only 0.25% of the design load size $\|f_1\|$; the new compliance value is 2.12047, which means an increase of 6%.

It is clear from the above example that small eigenvalues of the matrix $A(t)$ may give rise to instability of the truss: a small perturbation of the load may cause a large increase of the compliance. Thus we may conclude that for a stable truss the eigenvalues of the matrix $A(t)$ should be bounded well away from zero.

Recall from (19) that the bar-stiffness matrix $A(t)$ was given by

$$A(t) = B\,\mathrm{diag}\,(t)\,B^T = \sum_{\{v,w\}\in\mathcal{A}} t_{vw} b_{vw} b_{vw}^T = \sum_{i=1}^{n} t_i b_i b_i^T,$$

where B denotes the node-bar matrix of the truss and t the bar-volume vector of the truss. The equation (18) will have a unique solution only if $A(t) \succ 0$. This together with the above observations more than justifies the following assumption, which we assume to be satisfied in the sequel.

Assumption 3.3 *If $t_i > 0$ for $i = 1, \ldots, n$, then $A(t) \succ 0$.*

Note that this is also a physically meaningful assumption. It excludes rigid body motions of the ground structure: if all rigidities are positive then the potential energy stored by the structure under any nontrivial displacement is strictly positive.

3.2 Multi-load and robust versions of the TTD problem

In the previous sections we managed to derive a linear model for the TTD problem which turned out to be equivalent to the earlier proposed nonconvex quadratic model. A enlightening explanation of this very pleasant phenomenon is given in [8]. The explanation is based on a conic quadratic model of the TTD problem; this model

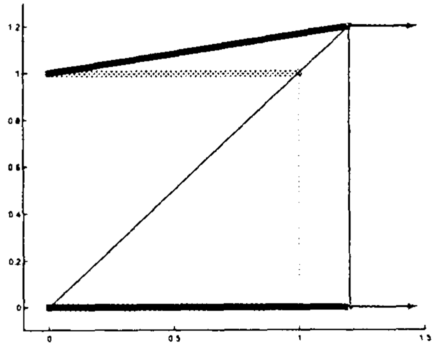

Figure 10: The optimal truss, unloaded (grey) and loaded (black).

The new compliance is then given by

$$\mathrm{Compl}_{f(\gamma)}(t) = \frac{1}{2}(f + \gamma g)^T \left(\Delta V + \frac{\gamma}{\lambda} g\right) \geq \mathrm{Compl}_f(t) + \frac{\gamma^2}{2\lambda}\|f\|^2 .$$

Here we used that $f^T g \geq 0$ (and hence also $g^T \Delta V \geq 0$) and $g^T g = \|f\|^2$. We see that the effect on the the compliance of such a perturbation of the load may be large if the eigenvalue is small. In our case, the bar-stiffness matrix $A(t)$ is given by

$$A(t) = B\,\mathrm{diag}\,(t) B^T = \frac{1}{2.001}\begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & -0.05 \\ 0 & 0 & 99.975 & 0.025 \\ 0 & -0.05 & 0.025 & 0.075 \end{bmatrix}. \tag{29}$$

Its smallest eigenvalue is 0.00548, and a corresponding eigenvector g such that $\|g\| = \|f\|$ and $f^T g \geq 0$ is given by

$$g = \|f\|\,(0.00000,\ 0.78819,\ -0.000154,\ 0.61544)^T .$$

From this we derive that

$$\mathrm{Compl}_{f(\gamma)}(t) \geq \mathrm{Compl}_f(t) + 18259\,\gamma^2 .$$

For $\gamma = 0.025$ (which amounts to a perturbation of 2.5%) the compliance becomes more than 13. For $\gamma = 0.0025$ the perturbed load becomes

$$f_2 = \begin{matrix} \\ 2 \\ \\ 4 \\ \\ \end{matrix}\left[\begin{array}{c} 10.000000 \\ 0.022867 \\ \hline 9.999995 \\ 0.031759 \end{array}\right]. \tag{30}$$

Figure 11 shows the loaded truss under f_2. The perturbation of the load is $\gamma\,\|g\|$,

Lemma 3.4 $\mathrm{Compl}_f(t) \le \tau$ *holds if and only if*

$$\tfrac{1}{2}\Delta V^T A(t)\Delta V - f^T\Delta V + \tau \ge 0, \quad \forall \Delta V.$$

Proof: By (32), $\mathrm{Compl}_f(t) \le \tau$ is equivalent to

$$\sup_{\Delta V \in \mathbf{R}^m} \left[f^T\Delta V - \tfrac{1}{2}\Delta V^T A(t)\Delta V \right] \le \tau.$$

Clearly, this holds if and only if the quadratic form

$$\tfrac{1}{2}\Delta V^T A(t)\Delta V - f^T\Delta V + \tau$$

is nonnegative for all $\Delta V \in \mathbf{R}^n$. □

Theorem 3.5 $\mathrm{Compl}_f(t) \le \tau$ *holds if and only if*

$$\begin{pmatrix} 2\tau & f^T \\ f & A(t) \end{pmatrix} \succeq 0.$$

Proof: This immediately follows from Lemma 3.4 and Lemma A.1. □

3.2.2 Semidefinite model for the multi-load TTD

The semidefinite representation of the compliance, as given by Theorem 3.5, enables us to formulate the TTD problem as a socalled semidefinite optimization problem:

$$\min_{\tau,t} \left\{ \tau \;:\; \begin{pmatrix} 2\tau & f^T \\ f & \sum_{i=1}^{n} b_i t_i b_i^T \end{pmatrix} \succeq 0, \quad \sum_{i=1}^{n} t_i \le w,\ t \ge 0 \right\}. \tag{33}$$

In the multi-load case we assume that that the set $\mathcal{F}$ of loading scenarios is a finite set,

$$\mathcal{F} = \{f_1, \ldots, f_k\}.$$

A big advantage of the above model is that it can be easily adapted to obtain a TTD that can withstand the loads f_i in $\mathcal{F}$ (not acting at the same time) in the best possible way.

$$\min_{\tau,t} \left\{ \tau \;:\; \begin{pmatrix} 2\tau & -f_j^T \\ -f_j & \sum_{i=1}^{n} b_i t_i b_i^T \end{pmatrix} \succeq 0,\ j = 1, \ldots, k,\ \sum_{i=1}^{n} t_i \le w,\ t \ge 0 \right\}. \tag{34}$$

The design variables are $t_i \in \mathbf{R}_+$, $i = 1, \ldots, n$, and $\tau \in \mathbf{R}$. Indeed, the linear matrix inequalities (LMI's) in (34) express the fact that the worst, over the loads $f_1, \ldots, f_k$, compliance of the construction yielded by the rigidities $t_1, \ldots, t_n$ does not exceed τ, while the linear constraints express the fact that $t = (t_1, \ldots, t_n)$ is an admissible design.

A crucial question is, of course, if we can solve these models efficiently! The answer is affirmative, as has become clear in Section 3.2.3 .

can be easily extended to the multi-load case. But only in the single-load case it naturally shrinks down to a linear model. This suggests that whenever a truss has to be designed that can withstand a finite set of (more than one) loads a linear model will not suffice.

In this section we consider the multi-load case, i.e. the case where we want to design a truss that is able to withstand a finite set of different loads in the best possible way. We show that an optimal truss can be obtained by solving a semidefinite minimization problem. Robust designs are obtained by allowing the set of loads to be infinitely large, but where the set is of a special form. Below we deal with ellipsoidal sets of loads. It will turn out that this case can also be modelled by a simple minimization problem, with only one semidefinite constraint.

The models in this section are based on a simple variational principle that will be introduced in the next section.

3.2.1 Variational Principle

Given a truss t and an external load f we define the *potential energy* $E_{t,f}(\Delta V)$ of the loaded system as a function of the displacement ΔV as follows:

$$\begin{aligned} E_{t,f}(\Delta V) &= \tfrac{1}{2}\,\Delta V^T A(t)\Delta V - f^T\Delta V \\ &= \tfrac{1}{2}\,\Delta V^T B\,\mathrm{diag}\,(t)\,B^T\Delta V - f^T\Delta V. \end{aligned} \tag{31}$$

Since $t \geq 0$, the matrix $A(t)$ is positive semidefinite, and hence $E_{t,f}(\Delta V)$ is a convex function. As a consequence, this function is bounded below if and only if its gradient vanishes for some ΔV, i.e., if and only of there exists ΔV such that

$$A(t)\,\Delta V = f.$$

Note that this is exactly equation (18), the equation for equilibrium. Thus we obtain the following Variational Principle:

> The equilibrium displacement of a truss t under an external load f is a minimizer of the quadratic form
>
> $$\tfrac{1}{2}\,\Delta V^T A(t)\Delta V - f^T\Delta V$$
>
> in the displacement ΔV; if this quadratic form is unbounded below, there is no equilibrium at all.

This is a typical variational principle in mechanics and physics. Such principles state that equilibria in certain physical systems occur at critical points (in good cases at minimizing points) of certain energy functionals. Variational principles are extremely powerful; in mechanical, electrical and other applications an issue of primary importance is to identify a tractable variational principle governing the model.

Note that in equilibrium, the (minimal) value of the energy function is given by $-\frac{1}{2}\,f^T\Delta V$. Thus, using (21), we obtain

$$-\mathrm{Compl}_f(t) = \min_{\Delta V}\left(\tfrac{1}{2}\,\Delta V^T A(t)\Delta V - f^T\Delta V\right). \tag{32}$$

As a consequence we have the following lemma, which needs no further proof.

For $\gamma = 0.01$ the perturbed load becomes

$$f_3 = \begin{array}{c} 2 \\ \\ 4 \\ \end{array} \left[\begin{array}{c} 10.000000 \\ -0.127949 \\ \hline 10.000192 \\ -0.059882 \end{array} \right]. \tag{36}$$

Figure 13 shows the loaded truss under f_3; The increase in the load size is $\gamma \|g\|$,

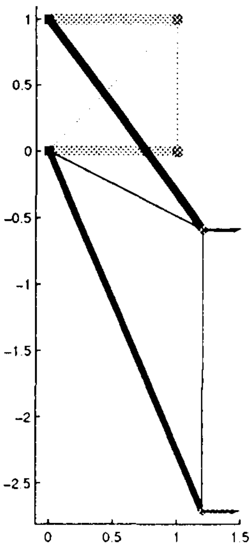

Figure 13: Instability of the multi-load truss w.r.t small occasional load.

which amounts to only 1% of the design load size $\|f\|$; the new compliance value is 2.25312, which means an increase of about 12.5%.

3.2.4 The robust TTD problem

We finally consider the socalled robust TTD problem, where we assume that the set of loads $\mathcal{F}$ is an ellipsoid:

$$\mathcal{F} = \left\{ f = Qu \ : \ u^T u \leq 1 \right\}, \quad Q \in \mathbf{M}^{m \times p}. \tag{37}$$

The matrix Q has to be chosen such that the ellipsoid $\mathcal{F}$ contains all possible loads that the truss has to withstand. Since the set $\mathcal{F}$ is infinite, we meet a difficulty not present in the case of finite $\mathcal{F}$, namely that the objective now is to minimize

$$\text{Compl}_{\mathcal{F}}(t) = \sup_{f \in \mathcal{F}} \text{Compl}_f(t), \tag{38}$$

which is the supremum of infinitely many SDR function. Fortunately, it is nevertheless easy to get an SDR for $\text{Compl}_{\mathcal{F}}(t)$.

3.2.3 Example of a multi-load truss

We turn back to the problem considered in Section 3.1.7. In that section we considered the single load truss for the design load f_1, and we also subjected the resulting truss to the load f_2, as given by (30). Thus these forces are given by

$$f_1 = \begin{array}{c} 2 \\ \\ 4 \end{array} \left[\begin{array}{c} 10.000000 \\ -0.005000 \\ \hline 10.000000 \\ 0.010000 \end{array} \right], \qquad f_2 = \begin{array}{c} 2 \\ \\ 4 \end{array} \left[\begin{array}{c} 10.000000 \\ 0.022867 \\ \hline 9.999995 \\ 0.031759 \end{array} \right]. \tag{35}$$

Using the same grid as there, we can now optimize the truss with respect to both loads by solving (34) for $k = 2$ and f_1 and f_2 as just given. The optimal multi-load truss t with respect to these loads, when loaded with one of these loads, behaves as depicted in Figure 12. The compliance of the truss with respect to f_1 is 2.012455,

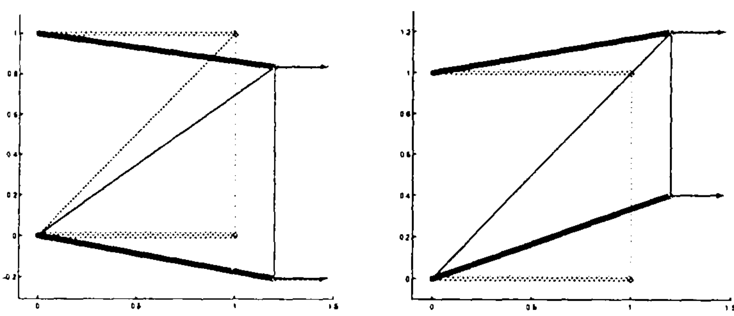

Figure 12: Multi-load truss subjected to the two separate design loads f_1 (left) and f_2 (right).

and with respect to f_2 is the compliance 2.015527 (which is also the optimal value of problem (34). Comparing the right figure in Figure 12 with the behavior of the single truss load under f_2, as depicted in Figure 11, we see that the new truss withstands this load considerably better.

Note that this result does not imply that the new truss is stable with respect to other small perturbations of the load. In fact, this is not the case. To obtain a perturbation for which the truss is most sensitive we use the same approach as in Section 3.1.7. For the new truss the smallest eigenvalue of $A(t)$ is $\lambda = 0.049163$ and the corresponding (normalized) eigenvector g such that $\|g\| = \|f_1\|$ and $f_1^T g \geq 0$ is given by

$$g = \|f_1\| \, (0.00000, \, -0.86938, \, 0.00135, \, -0.49413)^T .$$

Using the notation of Section 3.1.7, we replace the design load by

$$f(\gamma) = f_1 + \gamma g,$$

with $\gamma \geq 0$. Then the compliance with respect to $f(\gamma)$ satisfies

$$\mathrm{Compl}_{f(\gamma)}(t) \geq \mathrm{Compl}_{f_1}(t) + \frac{\gamma^2}{2\lambda} \|f_1\|^2 \approx \mathrm{Compl}_{f_1}(t) + 2034\,\gamma^2.$$

1		LO (1L)	SDO (2L)	SDO (3L)	SDO (Rob.)
2	t_{12}	0.4998	0.4981	0.4876	0.4576
3	t_{13}	0	0	0	0
4	t_{14}	0.0005	0.0054	0.0007	0.0377
5	t_{23}	0	0	0.0156	0.0377
6	t_{24}	0.0002	0.0011	0.0024	0.0093
7	t_{34}	0.4995	0.4954	0.4937	0.4576
8	Design Compl.	2.0020	2.0155	2.0519	2.1631
9	Compl. w.r.t f_1	2.0020	2.0125	2.0378	2.1548
10	Compl. w.r.t f_2	2.1205	2.0155	2.0155	2.1560
11	Compl. w.r.t f_3	3.7790	2.2531	2.0519	2.1609
12	$\lambda_{\min}^{-1}(A(t))$	182.5928	20.3405	17.2258	1.0820

Figure 14: Optimal trusses and their compliances.

both loads. The obtained truss turned out to be instable with respect to the load f_3, as defined by (36). The just mentioned results of these sections are summarized in the second and third column of the table in Figure 14.

The rows 2–7 give the volumes of the bars, row 8 the value of the compliance for the design, and rows 9–11 the actual compliance with respect to the loads f_1, f_2 and f_3 respectively. The last row gives the inverse of the smallest eigenvalue of the bar-stiffness matrix $A(t)$; we already have seen that this quantity can be considered as a measure for the robustness of the truss.

The fourth column in the table gives the corresponding values for the multi-load model (34) where the loads are now f_1, f_2 and f_3. It is depicted left in Figure 15 under the three given loads.

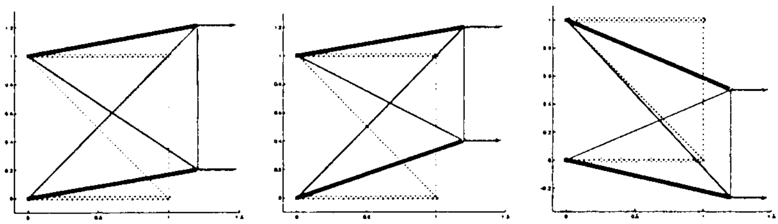

Figure 15: Multi-load truss loaded with f_1 (left), with f_2 (middle) and f_3 (right).

Finally, in the fifth column one finds the solution of the robust semidefinite model

Theorem 3.6 *Let $\mathcal{F}$ be as given by (37). Then one has one has* $\mathrm{Compl}_f(t) \leq \tau$ *for each $f \in \mathcal{F}$ if and only if*

$$\begin{pmatrix} 2\tau I_p & Q^T \\ Q & \sum_{i=1}^n b_i t_i b_i^T \end{pmatrix} \succeq 0.$$

Proof: With $\mathrm{Compl}_{\mathcal{F}}(t)$ as defined by (38), we may write

$$\begin{aligned} \mathrm{Compl}_{\mathcal{F}}(t) \leq \tau &\Leftrightarrow \tfrac{1}{2}x^T A(t)x - (Qu)^T x + \tau \geq 0,\ \forall x\ \forall (u\ :\ u^T u \leq 1) \\ &\Leftrightarrow \tfrac{1}{2}x^T A(t)x - (Qu)^T x + \tau \geq 0,\ \forall x\ \forall (u\ :\ u^T u = 1) \\ &\Leftrightarrow \tfrac{1}{2}x^T A(t)x - (Q\tfrac{u}{\|u\|})^T x + \tau \geq 0,\ \forall x\ \forall u \neq 0 \\ &\Leftrightarrow \tfrac{1}{2}(\|u\|\,x)^T A(t)(\|u\|\,x) - (Qu)^T(\|u\|\,x) + \tau u^T u \geq 0, \\ &\quad \forall x\ \forall u. \end{aligned}$$

Replacing $\|u\|\,x$ by $-y$ we obtain

$$\begin{aligned} \mathrm{Compl}_{\mathcal{F}}(t) \leq \tau &\Leftrightarrow 2\tau u^T u + 2u^T Q^T y + y^T A(t) y \geq 0, \quad \forall y\ \forall u \\ &\Leftrightarrow \begin{pmatrix} u \\ y \end{pmatrix}^T \begin{pmatrix} 2\tau I_p & Q^T \\ Q & A(t) \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} \geq 0, \quad \forall y\ \forall u \\ &\Leftrightarrow \begin{pmatrix} 2\tau I_p & Q^T \\ Q & A(t) \end{pmatrix} \succeq 0. \end{aligned}$$

For the last equivalence we used again Lemma A.1. This proves the theorem. □

Theorem 3.6 enables us to model the robust TTD problem as follows:

$$\min_{\tau,t} \left\{ \tau\ :\ \begin{pmatrix} 2\tau I_p & Q^T \\ Q & \sum_{i=1}^n b_i t_i b_i^T \end{pmatrix} \succeq 0,\quad \sum_{i=1}^n t_i \leq w,\ t \geq 0 \right\}. \tag{39}$$

This model finds the truss which is best able to withstand *all* the loads in the ellipsoidal set of loads

$$\mathcal{F} = \left\{ f = Qu\ :\ u^T u \leq 1 \right\}, \quad Q \in \mathbf{M}^{m \times p}.$$

Note that it does not tell us how to choose the matrix Q. But it is clear that we should choose Q in such a way that the ellipsoid $\mathcal{F}$ contains all loads that may occur.

3.2.5 Examples of robust designs

In this final section we consider again the 2×2 grid of Figure 9 in Section 3.1.7, which was also used in Section 3.2.3. In Section 3.1.7 we found the truss optimal with respect to f_1 by solving the linear model (24); it turned out that this truss is very unstable with respect to the load f_2 (cf. (35)). Subsequently, in Section 3.2.3, we used the multi-load semidefinite model (34) to find the optimal truss with respect to

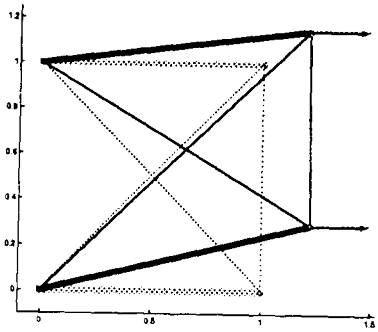

Figure 17: The robust truss w.r.t a small occasional load.

structure. Then (34) is a semidefinite problem with design dimension $n+1$, $n = O(M^2)$ being the number of tentative bars. The problem contains k (k is the number of loading scenarios) big LMI's (each of the row size $m+1$, where m is the number of degrees of freedom of the nodal set; $m \approx 2M$ for planar and $m \approx 3M$ for spatial trusses) and $n+1$ linear inequality constraints.

For a planar 15×15 grid with the left nodes fixed, we get $M = 225, n+1 = 25096, m = 420$. Even an LP problem with 25.000 variables should not be treated as a small one; a semidefinite problem of such a huge dimension is definitely not accessible for existing software.

The situation, however, is not hopeless, and the way to overcome the difficulty is offered by duality. The dual problems of (34) and (39) can be greatly simplified by analytical elimination of most of their variables. For example, the dual to the outlined multi-load truss problem can be converted to a semidefinite problem with nearly mk design variables; for the 15×15 ground structure and three scenarios, its design dimension is about 1300, which is within the range of applicability of existing solvers.

Below we will show how this can be reached for the multi-load problem; similar arguments can be applied to the robust problem. The outcome of the process will be summarized, for both cases, in Section 3.3.5. Note that (34) can be restated as

$$\min_{\tau,t}\left\{\tau : \begin{pmatrix} 2\tau & f_j^T \\ f_j & \sum_{i=1}^{n} b_i t_i b_i^T \end{pmatrix} \succeq 0,\ j=1,\ldots,k,\ \sum_{i=1}^{n} t_i \le w,\ t \ge 0\right\}. \tag{41}$$

The only change is that we replaced f_j by $-f_j$ in the LMIs; this makes no difference and is more convenient for our purpose.

3.3.1 Building the dual

We introduce dual matrix variables

$$\begin{pmatrix} \alpha_j & v_j^T \\ v_j & \beta_j \end{pmatrix} \succeq 0$$

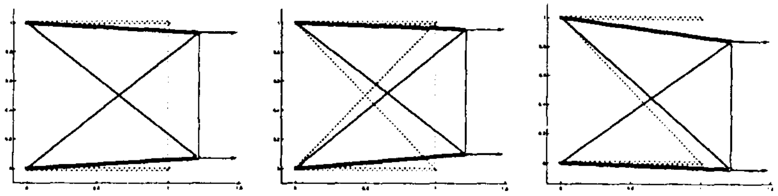

Figure 16: Robust truss loaded with f_1 (left), with f_2 (middle) and f_3 (right).

(39) for the matrix

$$Q = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 2 & 0 \\ 10 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

This means that the obtained truss is optimal with respect to the ellipsoidal set of loads

$$\mathcal{F} = \left\{ f = Qu = \begin{pmatrix} 10u_1 \\ 2u_2 \\ 10u_1 \\ 2u_3 \end{pmatrix} \;:\; u \in \mathbf{R}^3,\ u^T u \leq 1 \right\}.$$

It is depicted left in Figure 16 under the three given loads. ¿From the table it is clear that the inverse of the smallest eigenvalue of the bar-stiffness matrix is by far the smallest when compared with the other trusses. This means that the truss should be much more stable than the other trusses when loaded with any small occasional load. To verify this we conclude this section by loading this truss with a load of the form $f_1 + \gamma g$, where g is a unit eigenvector of the bar-stiffness matrix for its smallest eigenvalue. The smallest eigenvalue of $A(t)$ is $\lambda = 0.924217$ and the corresponding eigenvector g such that $\|g\| = \|f_1\|$ and $f_1^T g \geq 0$ is given by

$$g = \|f_1\| \,(0.01457,\ 0.70696,\ -0.01457,\ 0.70695)^T.$$

For $\gamma = 0.02$ the perturbed load becomes

$$f_4 = \begin{matrix} 2 \\ \\ 4 \end{matrix} \left[\begin{array}{c} 10.004122 \\ 0.194958 \\ \hline 9.995878 \\ 0.209958 \end{array} \right]. \tag{40}$$

Figure 17 shows the loaded truss under f_4; The increase in the load size is $\gamma \|g\|$, which amounts to 2% of $\|f_1\|$; the new compliance value is 2.19915.

3.3 Simplifying the semidefinite models by using duality

A disadvantage of the semidefinite models (34) and (39) is their huge dimension. Consider, for example, in the multi-load case (34) a truss with an M-node ground

Hence, the inequalities (a) and (d) imply the inequality

$$\sum_{j=1}^{k} b_i^T v_j \alpha_j^{-1} v_j^T b_i \le \gamma, \quad i = 1, \ldots, n.$$

On the other hand, if the last inequality holds, taking $\beta_j = v_j \alpha_j^{-1} v_j^T$, also (a) and (d) follow. Thus we conclude that the following problem has the same optimal value as (43).

$$\begin{array}{lrcll}
\text{supremum} & -2\sum_{j=1}^{k} f_j^T v_j - w\gamma & & & \\
\text{such that} & & & & \\
(b) & \gamma & \ge & 0, & \\
(c) & 2\sum_{j=1}^{k} \alpha_j & = & 1, & \\
(d') & \sum_{j=1}^{k} b_i^T v_j \alpha_j^{-1} v_j^T b_i & \le & \gamma, & i = 1, \ldots, n, \\
(e) & \alpha_j & > & 0, & j = 1, \ldots, k.
\end{array} \qquad (44)$$

Due to (e), and by the Schur complement lemma, the system of inequalities (d') is equivalent to the following system of LMIs:

$$\left(\begin{array}{ccc|c}
\alpha_1 & & & v_1^T b_i \\
& \ddots & & \vdots \\
& & \alpha_k & v_k^T b_i \\
\hline
b_i^T v_1 & \ldots & b_i^T v_k & \gamma
\end{array} \right) \succeq 0, \quad i = 1, \ldots, n, \qquad (45)$$

Consequently, we may replace the constraints (d') in problem (44) by (45). Note that (45) implies $\alpha_i \ge 0$, for all i. Hence, since our problem is still strictly feasible, omitting constraint (e) in (44) does not change the optimal value. Also note that (45) implies $\gamma \ge 0$, i.e. constraint (b) in (44). Thus we arrive at the final form of our dual problem of (41):

$$\begin{array}{ll}
\text{maximize} & -2\sum_{j=1}^{k} f_j^T v_j - w\gamma \\
\textit{such that} & \\
(a) & \left(\begin{array}{ccc|c}
\alpha_1 & & & v_1^T b_i \\
& \ddots & & \vdots \\
& & \alpha_k & v_k^T b_i \\
\hline
b_i^T v_1 & \ldots & b_i^T v_k & \gamma
\end{array} \right) \succeq 0, \; i = 1, \ldots, n, \\
(b) & 2\sum_{j=1}^{k} \alpha_j = 1,
\end{array} \qquad (46)$$

We already established that both the primal problem (41) and its dual problem (46) are strictly feasible. Consequently, both problems are solvable and their optimal values equal.

3.3.4 Back to primal

Problem (46) is not exactly the dual of (41) – it is obtained by elimininating part of the variables. What happens if we pass from (46) to its dual? It turns out that for the LMIs in (41), with $\alpha_j \in \mathbf{R}$, $\beta_j \in \mathbf{S}^m$, $v_j \in \mathbf{R}^m$, and dual scalar variables $\gamma \in \mathbf{R}_+$ for the weight constraint and $\tau_i \in \mathbf{R}_+$ for the nonnegativity constraints. The dual problem is then given by

$$
\begin{array}{llrcll}
\text{maximize} & -2\sum_{j=1}^k f_j^T v_j - w\gamma & & & & \\
\text{such that} & & & & & \\
(a) & \begin{pmatrix} \alpha_j & v_j^T \\ v_j & \beta_j \end{pmatrix} & & \succeq & 0, & j = 1, \dots, k, \\
(b) & & \gamma & \geq & 0, & \\
(c) & & \tau_i & \geq & 0, & i = 1, \dots, n, \\
(d) & & 2\sum_{j=1}^k \alpha_j & = & 1, & \\
(e) & & \sum_{j=1}^k b_i^T \beta_j b_i + \tau_i - \gamma & = & 0, & i = 1, \dots, n,
\end{array}
\tag{42}
$$

3.3.2 Eliminating the τ_i's

It is obvious that we can eliminate the slack variables τ_i, thus arriving at the equivalent problem

$$
\begin{array}{llrcll}
\text{maximize} & -2\sum_{j=1}^k f_j^T v_j - w\gamma & & & & \\
\text{such that} & & & & & \\
(a) & \begin{pmatrix} \alpha_j & v_j^T \\ v_j & \beta_j \end{pmatrix} & & \succeq & 0, & j = 1, \dots, k, \\
(b) & & \gamma & \geq & 0, & \\
(c) & & 2\sum_{j=1}^k \alpha_j & = & 1, & \\
(d) & & \sum_{j=1}^k b_i^T \beta_j b_i & \leq & \gamma, & i = 1, \dots, n,
\end{array}
\tag{43}
$$

It is worth noting that (43), and as a consequence also (42), is strictly feasible. Indeed, we may choose arbitrary positive reals α_j and by normalization, we may enforce (c). Choosing β_j large enough we enforce strict inequality in (a). Finally, choosing γ large enough also (d) will hold strictly. Since the primal problem (41) is also strictly feasible, we conclude that both problems have optimal solutions and that the optimal values are equal!

3.3.3 Eliminating the β_j's

Since (43) is strictly feasible, when adding the constraints $\alpha_j > 0$, for all j, the interior of the feasible region does not change, and hence neither does the optimal value. However, if $\alpha_j > 0$ then we may apply the Schur complement lemma to the constraints in (a), yielding the equivalent constraints

$$v_j \alpha_j^{-1} v_j^T \preceq \beta_j, \quad j = 1, \dots, k.$$

To prove the converse part of the theorem, let $(t_1, \ldots, t_n, \tau)$ be feasible to (41). As before we fix j, $1 \le j \le k$. For every $x \in \mathbf{R}$ the quadratic form (48) of $y \in \mathbf{R}^m$ is nonnegative, hence bounded below. The minimizer of this form satisfies

$$A(t)y = -xf_j \qquad \left[A(t) = \sum_{i=1}^{n} b_i t_i b_i^T\right]$$

and hence this equation is solvable for every x. This holds in particular if $x = -1$. Let the vector y_j satisfy

$$A(t)y_j = f_j.$$

Now define

$$q_i^j = t_i b_i^T y_j. \tag{49}$$

Then we have

$$\sum_{i=1}^{n} q_i^j b_i = \sum_{i=1}^{n} b_i t_i b_i^T y_j = A(t)y_j = f_j, \tag{50}$$

thus ensuring the validity of equation (c) in (47). It remains to show that the LMI's (a) in (47) are satisfied as well. Thus we finally need to show that for every $x \in \mathbf{R}$ and for every vector $\xi = (\xi_i)_{i=1}^n$ we have

$$F(x,\xi) \equiv 2\tau x^2 + 2\sum_{i=1}^{n} x q_i^j \xi_i + \sum_{i=1}^{n} t_i \xi_i^2 \ge 0. \tag{51}$$

Given x, let us set

$$\xi_i^* = -x b_i^T y_j,$$

and let us prove that the vector ξ^* minimizes $F(x,\xi)$. This is easy, because $F(x,.)$ is a convex quadratic form, and its partial derivative with respect to ξ_i at the point ξ_i^* is equal to (see (49))

$$2xq_i^j + 2t_i\xi_i^* = 2\left(xt_i b_i^T y_j - t_i x b_i^T y_j\right) = 0,$$

for all i, proving the claim. Thus, to complete the proof of (51), we only need to show that $F(x,\xi^*) \ge 0$. This goes as follows:

$$\begin{aligned}
F(x,(\xi_i^*)_{i,s}) &= 2\tau x^2 + 2\sum_{i=1}^{n} x q_i^j \xi_i^* + \sum_{i=1}^{n} t_i {\xi_i^*}^2 \\
&= 2\tau x^2 - 2\sum_{i=1}^{n} x^2 q_i^j b_i^T y_j + \sum_{i=1}^{n} x^2 y_j^T b_i t_i b_i^T y_j \\
&= 2\tau x^2 - 2x\left(\sum_{i=1}^{n} q_i^j b_i\right)^T x y_j + x^2 y_j^T A(t) y_j \\
&= 2\tau x^2 - 2x f_j^T x y_j + x^2 y_j^T A(t) y_j.
\end{aligned}$$

The last reduction used (50). Hence we write

$$F(x,\xi^*) = \begin{pmatrix} x \\ -xy_j \end{pmatrix}^T \begin{pmatrix} 2\tau & f_j^T \\ f_j & A(t) \end{pmatrix} \begin{pmatrix} x \\ -xy_j \end{pmatrix}.$$

Since $(t_1, \ldots, t_n, \tau)$ is feasible to (41) the last expression is nonnegative, and hence the proof is complete. □

we end up with a nontrivial (and instructive) equivalent formulation of (41), namely, with the problem

$$\begin{array}{llcll}
\min & \tau & & & \\
\text{s.t.} & & & & \\
(a) & \left(\begin{array}{c|ccc} 2\tau & q_1^j & \dots & q_n^j \\ \hline q_1^j & t_1 & & \\ \vdots & & \ddots & \\ q_n^j & & & t_n \end{array}\right) & \succeq & 0, & j=1,\dots,k, \\
(b) & \sum_{i=1}^n t_i & \le & w, & \\
(c) & \sum_{i=1}^n q_i^j b_i & = & f_j, & j=1,\dots,k,
\end{array} \tag{47}$$

where the design variables are

- $t_i \in \mathbf{R}_+$ and $\tau \in \mathbf{R}$;
- $q_i^j \in \mathbf{R}$, $j = 1, \dots, k$, $i = 1, \dots, n$.

(47) is not the straightforward dual of (46); it is obtained from this dual by eliminating part of the variables. Instead of deriving (47) in this way, we prefer to give a direct proof of its equivalence to (41) by proving the following result.

Theorem 3.7 *A collection $(t_1, \dots, t_n, \tau)$ is feasible to (41) if and only if it can be extended by properly chosen*

$$\left\{q_i^j \in \mathbf{R}^p \;:\; j=1,\dots,k,\, i=1,\dots,n\right\}$$

to a feasible solution to (47).

Proof: Let $(t_1, \dots, t_n, \tau)$ and

$$\left\{q_i^j \in \mathbf{R}^p \;:\; j=1,\dots,k,\, i=1,\dots,n\right\}$$

compose a feasible solution to (47). Fixing j $(1 \le j \le k)$, we should prove the validity of the LMIs in (41). Thus we should prove that for every pair (x, y) with $x \in \mathbf{R}$ and $y \in \mathbf{R}^m$ we have

$$2\tau x^2 + 2x f_j^T y + y^T \left(\sum_{i=1}^n b_i t_i b_i^T\right) y \ge 0. \tag{48}$$

In view of (c) in (47) the left hand side of (48) is equal to

$$2\tau x^2 + 2x \sum_{i=1}^n q_i^j b_i^T y + y^T \left(\sum_{i=1}^n b_i t_i b_i^T\right) y = 2\tau x^2 + 2\sum_{i=1}^n x q_i^j \xi_i + \sum_{i=1}^n t_i \xi_i^2,$$

where $\xi_i = b_i^T y$. The resulting expression is nothing but the value of the quadratic form with the matrix from the left-hand side of the LMI (a) in (47) at the vector comprised of x and $(\xi_i)_i$, and therefore is nonnegative, as claimed.

with M free nodes. Note that in this case $p = 1$, $n \approx 0.5M^2$ and $m \equiv 2M$. Assuming $k \ll M$, here are the sizes of (41), (46) and (47) :

	Design dimension
(41)	$n + 1 \approx 0.5M^2$
(46)	$mk + k + 1 \approx 2kM$
(47)	$nk + n + 1 \approx 0.5kM^2$

	# and sizes of LMI's
(41)	k of $(2M + 1) \times (2M + 1)$
(46)	$n \approx 0.5M^2$ of $(k + 1) \times (k + 1)$
(47)	k of $(n + 1) \times (n + 1)$

	# of linear constraints
(41)	$n + 1 \approx 0.5M^2$
(46)	1
(47)	$kM + 1$

We see that if the number k of loading scenarios is small (which normally is the case), the design dimension of the dual problem (46) is by orders of magnitude less than the design dimensions of both primal problems. As a kind of penalization, the dual problem involves a lot ($\approx 0.5M^2$) of LMI's instead of just k LMI's in the primal problems. But the LMI's in the primal problems are large, and these in the dual small in size. When solving these problems with the best-known numerical techniques so far (the interior-point algorithms), the computational effort for (41) is $O(M^6)$, while for (46) it is only $O(k^3M^3)$. For large M and small k this does make a significant difference!

Of course, there is an immediate concern about the dual problem: the actual design problems are not seen in it at all. How do we recover a (nearly) optimal construction from a (nearly) optimal solution to the dual problem? In fact, however, there is no reason to be concerned: the required recovering routines exist and are cheap computationally.

4 Concluding remarks

In this paper we illustrated the use of conic optimization as a powerful tool for the mathematical modelling of inherently nonlinear problems. As an example we used the truss topology design problem. One may check the reference list below to observe that with the exception of one paper all relevant papers appeared in the last 10 years. Indeed, the subject thanks its existence to the development of efficient solution methods for conic optimization problems in the last decade. Especially the possibility of modelling robustness of a design in a computationally tractable way opens the way to many new applications. We demonstrated this only for the TTD problem, which is a popular application in the literature. For other interesting applications we refer to [8] and the other references. It may be expected that the ongoing research will bring forth many new important applications in the near future.

3.3.5 Summary of the semidefinite models for multi-load and robust TTD

In this section we summarize the results of the previous sections by presenting explicit forms of the primal (41), the simplified dual (46) and the simplified primal problem (47) for the multi-load problem, and similar versions for the robust truss design problems, respectively.

Multi – load TTD	**Robust TTD**
Primal problem	
$\min \tau$ s.t. $(a)\quad \begin{pmatrix} 2\tau & f_j^T \\ f_j & \sum_{i=1}^n b_i t_i b_i^T \end{pmatrix} \succeq 0,\ j = 1:k,$ $(b)\quad \sum_{i=1}^n t_i \le w,$ $(c)\quad t_i \ge 0,\ i = 1:n.$	$\min \tau$ s.t. $(a)\quad \begin{pmatrix} 2\tau I_p & Q^T \\ Q & \sum_{i=1}^n b_i t_i b_i^T \end{pmatrix} \succeq 0,$ $(b)\quad \sum_{i=1}^n t_i \le w,$ $(c)\quad t_i \ge 0,\ i = 1:n.$
Simplified dual problem	
$\max -2\sum_{j=1}^k f_j^T v_j - w\gamma$ s.t. $(a)\quad \left(\begin{array}{ccc\|c} \alpha_1 & & & v_1^T b_i \\ & \ddots & & \vdots \\ & & \alpha_k & v_k^T b_i \\ \hline b_i^T v_1 & \ldots & b_i^T v_k & \gamma \end{array}\right) \succeq 0,\ i = 1:n,$ $(b)\quad 2\sum_{j=1}^k \alpha_j = 1.$	$\max -2\mathrm{Tr}\,(Q^T V) - w\gamma$ s.t. $(a)\quad \begin{pmatrix} \alpha & V^T b_i \\ b_i^T V & \gamma \end{pmatrix} \succeq 0,\ i = 1:n,$ $(b)\quad 2\mathrm{Tr}\,(\alpha) = 1.$ $\alpha \in \mathbf{R}^p,\ V \in \mathbf{M}^{m\times p}$
Simplified primal	
$\min \tau$ s.t. $(a)\quad \left(\begin{array}{c\|ccc} 2\tau & q_1^j & \ldots & q_n^j \\ \hline q_1^j & t_1 & & \\ \vdots & & \ddots & \\ q_n^j & & & t_n \end{array}\right) \succeq 0,\ j = 1:k,$ $(b)\quad \sum_{i=1}^n t_i \le w,$ $(c)\quad \sum_{i=1}^n b_i q_i^j = f_j,\ j = 1:k.$	$\min \tau$ s.t. $(a)\quad \left(\begin{array}{c\|ccc} 2\tau I_p & q_1 & \ldots & q_n \\ \hline q_1^T & t_1 & & \\ \vdots & & \ddots & \\ q_n^T & & & t_n \end{array}\right) \succeq 0,$ $(b)\quad \sum_{i=1}^n t_i \le w,$ $(c)\quad \sum_{i=1}^n b_i q_i^T = Q.$

3.3.6 Evaluation

To understand how fruitful our effort was, it is enlightening to compare the sizes of the original problem (41) and the reformulation (46) of its simplified dual problem. Let us restrict ourselves to the simple case of a planar k-load truss design problem

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A Appendix

Lemma A.1 (Shor [20]) *Let $A \in \mathbf{R}^{n\times n}, b \in \mathbf{R}^n$ and $c \in \mathbf{R}$. Then the quadratic form $x^T Ax + 2b^T x + c$ is nonnegative for all $x \in \mathbf{R}^n$ if and only if*

$$\begin{pmatrix} A & b \\ b^T & c \end{pmatrix} \succeq 0 \text{ or, equivalently } \begin{pmatrix} A & -b \\ -b^T & c \end{pmatrix} \succeq 0.$$

Proof: The proof consists of a sequence of logically equivalent statements, as follows:

$$\begin{aligned} \forall x \;:\; x^T Ax + 2b^T x + c \geq 0 \quad &\Leftrightarrow \\ \forall (t \neq 0, x) \;:\; t^{-2} x^T Ax + 2t^{-1} b^T x + c \geq 0 \quad &\Leftrightarrow \\ \forall (t \neq 0, x) \;:\; x^T Ax + 2tb^T x + ct^2 \geq 0 \quad &\Leftrightarrow \\ \forall (t, x) \;:\; x^T Ax + 2tb^T x + ct^2 \geq 0 \quad &\Leftrightarrow \\ \forall (t, x) \;:\; \begin{pmatrix} x \\ t \end{pmatrix}^T \begin{pmatrix} A & b \\ b^T & c \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \geq 0 \quad &\Leftrightarrow \quad \begin{pmatrix} A & b \\ b^T & c \end{pmatrix} \succeq 0. \end{aligned}$$

□

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